COMPUTER AIDED GEOMETRIC METHODS FOR DEVELOPMENT OF CERTAIN CLASSES OF RULED SURFACES

by

B. GURUNATHAN

DEPARTMENT OF MECHANICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

FEBRUARY, 1986

COMPUTER AIDED GEOMETRIC METHODS FOR DEVELOPMENT OF CERTAIN CLASSES OF RULED SURFACES

A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

by

B. GURUNATHAN

to the

DEPARTMENT OF MECHANICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

FEBRUARY, 1986

Dedicated

to

My Parents

Balasubramaniam,P.S. and Muthulakshmi,B

CERTIFICATE

Certified that this work on 'COMPUTER AIDED GEOMETRIC METHODS FOR DEVELOPMENT OF CERTAIN CLASSES OF RULED SURFACES' has been carried out under my supervision and that it has not been submitted elsewhere for a degree.

Dhande

10th February, 1986

(S.G. Dhande)
Professor of Mechanical Engineering
and Computer Science and Engineering
Indian Institute of Technology
Kanpur

ACKNOWLEDGEMENT

I am very much indebted to Dr. S.G. Dhande for his expert guidance and continued encouragement during the course of this work. I remember the encouragement and the warmth of his kindness that I have enjoyed during the period of my research programme right from the very first day. I remain grateful to him forever.

I am thankful to many members of Faculty of Department of Mechanical Engineering and other departments who have encouraged and helped me a lot.

I am very much thankful to the Government of Tamilnadu and the Director of Technical Education, Tamilnadu for having sponsored me for the Ph. D. programme under Quality Improvement Programme.

I thank the Faculty and Staff of Computer Centre, I.I.T., Kanpur for the excellent facilities provided in the form of DEC-1090 system and other graphics equipments and for the easy access to them.

Good friends make the life enjoyable. To all my friends who shared and stood by me during days of happiness as well as difficulties, I am thankful.

I thank Mr. Pandimani, Mr. Sukamal Ghosh, Mr. Allesu Kanjirathinkal for the help rendered during the preparation of this thesis.

Mr. D.P. Saini has done neat typing of this thesis. Mr. S.S. Kushwaha and Mr. G.K. Shukla have done good tracing of the figures. I am thankful to them.

I thank my wife Umarani and children Saravanakumar, Rajkumar and Ganeshkumar for their patient understanding and the refreshingly different atmosphere presented in the home.

February, 1986 B. Gurunathan

CONTENTS

LIST OF FIGURES

LIST OF TABLES

NOMENCLATURE

a	-	semi-major diameter of a super-ellipse
a_{i0}	-	semi-major diameter of the elliptical base of a conical helical convolute at the starting point
b	-	semi-major diameter of a super-ellipse
$\underline{b}$	-	bi-normal vector
b_{i0}	-	semi-minor diameter of the elliptical base of a conical helical convolute at the starting point
c_i	-	axial movement of the generic point along the helix per unit radian of rotation
$c_{kd,kt}$	-	distance of the kt^{th} surface from the mean surface in the kd^{th} thick surface
$D_{kd,kt}$	-	kt^{th} directrix on the end surface E_{kd}
$\underline{d}$	-	direction vector of a straight line directrix
d	-	distance of the centre of a super-ellipse from the origin
E_{kd}	-	kd^{th} end surface of the multiple thick surfaces in series
e_i	-	rate at which the semi-major (or minor) diameter of a conical helix reduces per radian of rotation per unit value of the semi-major (or minor) diameter

$\underline{g}$	-	unit directional vector of a straight line generatrix
h	-	thickness of a thick surface
$\underline{K}$	-	curvature
$\underline{K}_g$	-	Geodesic curvature
$\underline{K}_n$	-	normal curvature
k	-	magnitude of curvature
k_g	-	magnitude of geodesic curvature
L	-	length of the generatrix
$\underline{N}$	-	normal
n	-	power index of a super-ellipse
$\underline{n}_c$	-	unit normal to a curve
$\underline{n}_d$	-	unit normal to the director
$\underline{n}_s$	-	unit normal to a surface
$\underline{n}_{o_{kt}}$	-	unit normal to the development plane of the kt^{th} surface of a thick surface
$\underline{r}$	-	position vector of a point (on a curve)
$\dot{\underline{r}}$	-	first derivative of $\underline{r}$ with respect to the parameter of the curve
$\ddot{\underline{r}}$	-	second derivative of $\underline{r}$ with respect to the parameter of the curve
$S_{kd,kt}$	-	kt^{th} surface in the kd^{th} thick surface
s	-	arc length of a curve
s_h	-	number of slices into which each half of a thick surface is to be divided

$[T]$ - transformation matrix corresponding to a directrix of a super-conical convolute

TS_{kd} - kd^{th} thick surface

$\underline{t}$ - unit tangent vector

$\underline{t}_{o_{kt}}$ - unit tangent vector to the primary directirx of the kt^{th} surface of a thick surface

(x_d , y_d) - coordinate values of a point in the development of a surface

$(x_{d_{kt}}, y_{d_{kt}}, z_{d_{kt}})$ - coordinate values of a point in the development of the kt^{th} surface of a thick surface before stacking

$(x_{dm_{kt}}, y_{dm_{kt}}, z_{dm_{kt}})$ - coordinate values of a point in the development of the kt^{th} surface of a thick surface after stacking

α , β - angular parameters of the directional vector of the generatrix of a ruled surface, class-I

α, β, γ - angles specifying the spatial configuration of a super-ellipse

Σ - a bi-parametric surface

σ - tangent plane of a bi-parametric surface

$\emptyset$ - angle between the arc-tangent and the generatrix

Ψ - arc-tangent angle

Subscripts

Subscript		indicate
a	-	centre line of a duct
c	-	a curve
cs	-	cross section of a duct
i	-	primary directrix of a ruled surface
j	-	secondary directrix of a ruled surface
kd	-	serial number of the conical convolute of a thin duct
	-	serial number of a thick surface in the multiple thick surfaces in series
kt	-	serial number of a surface of a thick surface
s	-	surface
		parameters
q,u,w,θ	-	parameters of a curve
v	-	parameter of the generatrix
u,v	-	parameters of a bi-parametric surface

COMPUTER AIDED GEOMETRIC METHODS FOR DEVELOPMENT OF CERTAIN CLASSES OF RULED SURFACES

A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

By
B. GURUNATHAN

to the
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

SYNOPSIS

Geometrical design constitutes the first and foremost portion of the design specification of any object. A major class of objects to be designed are different types of surfaces. In some methods of fabrication of surfaces from sheet metal, it is necessary to find the development of the surface. Surfaces can be classified as (i) developable surfaces and (ii) non-developable surfaces. Single-curved ruled surfaces are developable. Warped ruled surfaces, doubled-curved surfaces and free-form surfaces are non-developable.

Mapping of one surface onto another observing certain conditions of mapping is an area of study in differential geometry. If a surface Σ_1 is mapped onto a planar surface Σ_2 isometrically on one-to-one basis such that the mapping is isogonal as well as isoareal then the mapping Σ_2 is the development of the surface Σ_1. Using this condition and the fact that the geodesic curvature of a curve lying on the surface Σ_1 is equal to the planar curvature of the development image of this curve in surface Σ_2, it is shown in the present work that suitable computational algorithms for obtaining the development of the curved surface can be formulated.

Detailed mathematical models and computational algorithms have been proposed for the development of the following types of surfaces.

(i) Helical convolutes-cylindrical as well as conical,

(ii) Conical convolutes,

(iii) Ducts-planar as well as spatial and

(iv) Thick surfaces-single surfaces as well as multiple surfaces in series.

Chapter 1 of the thesis reviews the existing graphical methods for developing surfaces. It also contains a section on classification of surfaces into ruled, double-curved and free-form surfaces. For each

category, several illustrative examples have been cited and mathematical representation in parametric form for each of these surfaces has also been given.

In Chapter 2 it has been shown that a developable surface can be defined as an envelope of a family of one-parameter planar surfaces. The conditions of mapping a curved developable surface isometrically onto a planar surface have been discussed. The geodesic curvature of a spatial curve and the Serret-Frenet equations are used in the process of mapping. Finally separate algorithms are given in a general manner for the development of (a) ruled surfaces with single directrix and (b) ruled surfaces with two directrices.

A major class of single curved ruled surfaces is the conical convolute surfaces. Suitable mathematical expressions to develope such ducting surfaces are given in Chapter 3. A general form of conical convolute, called super-conical convolute, has been defined. Development of conical convolutes connecting end section's of the following shape are presented as case studies: (a) circle-circle, (b) circle-ellipse, (c) ellipse-ellipse and (d) super-ellipse-super-ellipse.

The development of helical convolutes-cylindrical as well as conical - is presented in Chapter 4. The base

of the helix can be either a circle or an ellipse. A close-form solution is given for the development of the cylindrical helical convolute of circular base. Three examples of development of helical convolutes are given at the end of this chapter.

In Chapter 5, the development of thin ducts is considered. Thin ducts are approximated as a set of super conical convolutes in series, whose shape, size and spatial configuration can be controlled by the parameter of the centreline of the duct. Necessary equations and a suitable algorithm for the development of a thin duct are given. Development of (a) a planar duct of variable circular cross section and (b) a spatial duct of cross section varying in size and shape (from circle to a super-ellipse) are presented as examples.

Development of uniformly thick surfaces is given in Chapter 6. The thick surface is modeled to be a set of thin surfaces. A mean surface is defined. If the mean surface is a developable one, the other surfaces in the set are shown to be developable. Suitable mathematical model is given for the development of these individual surfaces and then for getting the development of the thick surface. Single thick surface as well as multiple thick surfaces in series are considered and

suitable algorithms for their development are given. Finally an example is given for the development of a thick surface.

Chapter 7 summarizes the results given in the present work. Suggestions for further work have also been indicated in this chapter.

Chapter 1

INTRODUCTION

1.1 Introduction

Engineering design is a methodology that incorporates iterative procedures of analysis and synthesis for components as well as systems [1]. Design specifications of an object generally consist of geometrical parameters, material properties and manufacturing details. It has been observed that geometrical design constitutes the first and foremost portion of design specifications for any object. Consequently a large number of literature/information is available for geometrical design using manual drafting procedures [2-8].

Though the manual drafting procedures have helped designers over the past several years, these techniques have certain drawbacks. The procedures are time consuming. If the geometry of an object to be designed is in the form of complicated shapes, then defining the geometry as well as displaying it is extremely difficult. The manual drafting procedures are also dependent on the accuracy of drafting instruments and the skill of the draftsman.

Considerable advances have been made in the past two decades in the area of computer-controlled drafting and computational aids [9-13]. These aids are in the form of graphics terminals and plotters, digitizers, time-shared interactive computers and a set of input devices. With such accessories a designer is now able to mathematically model the geometry of an object, represent it in computers and after suitable manipulations display it effectively on graphics devices. The advances made in the use of computers in geometric design have given rise to the field of Computer Aided Geometric Design [1, 14-16].

One of the major class of objects to be designed are different types of surfaces to be fabricated out of sheet metal. To fabricate these surfaces the sheet metal is to be cut to the required shape and size and then it should be folded and bent along specified lines or formed so as to get the required surface. This requires that the surface be unfolded and laid out into a plane first. This process is called the development of a surface.

In the following section, different types of surfaces are discussed. These can be classified as (i) developable surfaces and (ii) non-developable surfaces. Developable surfaces are surfaces which can be fabricated

from a sheet metal without causing any appreciable plastic deformation during manufacture. Plane and single curved ruled surfaces belong to this category. These surfaces can be developed exactly. Warped ruled surfaces, double-curved surfaces and free-form surfaces are non-developable surfaces and are fabricated from sheet metal by forming processes which introduce appreciable plastic deformations. In this case the development of a surface can be found out only approximately. Plastic deformations induced during manufacturing process are also required to be taken into account while obtaining the shape and size of the blank.

1.2 Mathematical Definition of Surfaces

Three dimensional surfaces can broadly be classified as

(i) ruled surfaces,

(ii) double-curved surfaces and

(iii) free-form or sculptured surfaces.

A detailed classification of different types of surfaces belonging to the above mentioned categories are available in books on Geometry, Differential Geometry and Computational Geometry [2-8, 12, 13, 17-24]. In the following sections parametric representation of some of these surfaces are given.

1.2.1 Ruled Surfaces

When a straight line, called the generatrix, moves in space while being in contact with one or more straight or curved lines, called the directrices so as to form a surface then a ruled surface is obtained. Some times the generatrix moves such that not only it is in contact with the directrix/directrices but also remain parallel to a plane called director. Ruled surfaces can further be classified as:

(i) single-curved surface and

(ii) warped surfaces.

Single-curved ruled surfaces have curvature in one direction only. They are developable. Warped surfaces are not developable as adjacent positions of the generatrix are not in a plane.

The following nomenclature is used in the representation of surfaces.

$\underline{r}_1$, $\underline{r}_2$, $\underline{r}_3$	-	directrices
u , w , q	-	parameters of directrices
$\underline{d}$	-	directional vector of a straight line directrix
$\underline{n}_d$	-	normal to a director
$\underline{t}_1$, $\underline{t}_2$	-	unit tangent to the directrices

$\underline{n}_1$, $\underline{n}_2$ - unit principal normal to the directrices

$\underline{b}_1$, $\underline{b}_2$ - unit bi-normal to the directrices

$\underline{g}$ - directional vector of the straight line generatrix

v - parameter along the generatrix

u , v - parameters of the surface

Unless otherwise specified the range for u, w, q and v are $u_1 \leq u \leq u_2$, $w_1 \leq w \leq w_2$, $q_1 \leq q \leq q_2$ and $0 \leq v \leq 1$. Wherever the parameters w and q appear, they are shown to be functions of u through the given conditions.

I. Planar Surface

The parametric representation is

$$\underline{n} \cdot \underline{r} = \delta \qquad (1.1)$$

where $\underline{n}$ is the normal to the plane.

II. Single-curved Surfaces

(a) Cylinders

The parametric representation is

$$\underline{r}\,(u\,,\,v) = \underline{r}_1\,(u) + v\,\underline{g} \qquad (1.2)$$

$$0 \leq v \leq h$$

If $\underline{r}_1$ (u) is a planar curve and $\underline{g} \times \underline{b}_1 = 0$ then it is a right cylinder. If $\underline{g} \times \underline{b}_1 \neq 0$, then it is an oblique cylinder.

(b) Cones

Cones are represented parametrically by

$$\underline{r}\,(u\,,\,v) = (1 - v)\,\underline{r}_1\,(u) + v\,\underline{r}^{(V)} \qquad (1.3)$$

where $\underline{r}^{(V)}$ is the position vector of the vertex of the cone. If $\underline{r}_1$ (u) is a planar curve and if $(\underline{r}^{(V)} - \underline{r}^{(C)}) \times \underline{b}_1 = 0$ it is a right cone. If $(\underline{r}^{(V)} - \underline{r}^{(C)}) \times \underline{b}_1 \neq 0$, then it is an oblique cone. Here $\underline{r}^{(C)}$ is the position vector of the centre of the base of the cone.

(c) Convolutes

The parametric representation is

$$\underline{r}\,(u\,,\,v) = (1 - v)\underline{r}_1(u) + v\,\underline{r}_2\,(w) \qquad (1.4)$$

where $\underline{r}_1$ (u) and $\underline{r}_2$ (w) are open curves. The condition to be satisfied is

$$\frac{d\underline{r}_1\,(u)}{du} \times \frac{d\underline{r}_2\,(w)}{dw} \cdot (r_2\,(w) - r_1\,(u)) = 0 \qquad (1.5)$$

For conical convolutes $\underline{r}_1$ (u) and $\underline{r}_2$ (w) are closed curves. Helical convolutes are defined by

$$\underline{r}\,(u\,,\,v) = \underline{r}_1\,(u) + v\underline{t}_1\,(u) \qquad (1.6)$$

where $\underline{r}_1$ (u) is a helix ; $-\infty \leq v \leq \infty$

III. Warped Surfaces

Warped surfaces are of two groups, namely (a) single-ruled surfaces and (b) double-ruled surfaces.

(a) Single-ruled Surfaces

The parametric representation is

$$\underline{r}\,(u\,,\,v) = (1 - v)\,\underline{r}_1\,(u) + v\,\underline{r}_2\,(w) \qquad (1.7)$$

subject to the condition

$$(\,\underline{r}_2\,(w) - \underline{r}_1\,(u))\,.\,\underline{n}_d = 0 \quad . \qquad (1.8)$$

If $\underline{r}_1$ (u) and $\underline{r}_2$ (w) are curves then it is a cylindroid. In the case of cow's horn, the above given condition (Eqn. (1.3))is not applicable since no director is involved in the generation of the surface. The conditions applicable for cow's horn are

$$\{\,\underline{r}_2\,(w) - \underline{r}_1\,(u)\}\, x\{\,\underline{r}_3\,(q) - \underline{r}_1\,(u)\} = 0$$

and

$$\{\,\underline{r}_1\,(u) - \underline{r}_3\,(q)\}\, x\{\,\underline{r}_2\,(w) - \underline{r}_3\,(q)\} = 0$$

$$\ldots \qquad (1.9)$$

where $\underline{r}_3$ (q) is the position vector of the point on the straight line directrix where the generatrix intersects the straight line directrix. From Eqns. (1.9), the parameters w and q are given as functions of u.

If $\underline{r}_1$ (u) is a curve and $\underline{r}_2$ (w) is a straight line conoids are obtained. Let the direction vector of $\underline{r}_2$ (w) be $\underline{d}$. Then if $\underline{d}$ x $\underline{n}_d = 0$, it is a right conoid ; otherwise it is an oblique conoid.

The parametric equation and conditions for warped cone are the same as that for the cow's horn.

If $\underline{r}_1$ (u)is a helix and $\underline{r}_2$ (w) is a straight line directrix then helicoidal forms of single-ruled warped surfaces are obtained. If $\underline{r}_1$ (u) is a cylindrical helix and if $\underline{d} \times \underline{n}_d = 0$, then it is a right cylindrical helicoid. If $\underline{r}_1$ (u) is a cylindrical helix and if the director is a cone whose normal is given by $\underline{n}_d$ (u) such that $\underline{d} \cdot \underline{n}_d (u) = \text{constant}$, then it is an oblique cylindrical helicoid. If $\underline{r}_1$ (u) is a conical helix instead of cylindrical helix then a right conical helicoid is obtained if $\underline{d} \times \underline{n}_d = 0$; if $\underline{d} \cdot \underline{n}_d (u) = \text{constant}$ an oblique conical helicoid is obtained.

(b) Double-ruled Surfaces

(i) Circular form : Hyperboloid of revolution.

If the axis of the hyperboloid is taken as the z axis, the parametric equation of the surface is given by

$$\begin{bmatrix} \|\underline{a}\| & \cos (u + \phi) \\ \|\underline{a}\| & \sin (u + \phi) \\ \{\underline{r}^{(L)} & + v\underline{g}\} \cdot \underline{d} \end{bmatrix} \qquad (1.10)$$

where

$$\underline{a} = \{\underline{r}^{(L)} + v\underline{g}\} - \{(\underline{r}^{(L)} + v\underline{g}) \cdot \underline{d}\}\, \underline{d}$$

$$\cos \phi = \frac{\underline{a} \cdot \underline{x}}{\|\underline{a}\|}$$

where $\underline{d}$ is the direction vector of the axis and $\underline{x}$ is the unit vector along the x axis. The vectors $\underline{r}^{(L)}$ and $\underline{g}$ are the position vector of the end point on the generatrix corresponding to $v = 0$ and the direction vector of the generatrix, when the generatrix is at its initial position.

(ii) Elliptical form : Elliptical hyperboloid.

The parametric equation and conditions are the same as Eqns. (1.7) and (1.9). Here $\underline{r}_1$ (u), $\underline{r}_2$ (w) and $\underline{r}_3$ (q) define the three elliptical directrices.

(iii) Parabolic form : Parabolic hyperboloid

The parametric equation and conditions are the same as Eqns. (1.7) and (1.8). Here $\underline{r}_1$ (u) and $\underline{r}_2$ (w) define the two straight line directrices.

1.2.2 Double-curved Surfaces

If a constant or variable generating curve is moved along another curve then a double-curved surface is obtained. Double-curved surfaces are classified as (i) surfaces of revolution and (ii) surfaces of general form.

I. Surfaces of revolution are obtained when a generating curve revolves about an axis (a straight line). If the generating curve and the axis are co-planar then the

surface generated conforms to the curve of the generatrix. If the axis of the surface coincides with the z axis and if a generic point on the generatrix, when it is stationary, is given by $\underline{r}_c\ (v)$, then corresponding generic point on the surface is given by

$$\underline{r}\ (u\ ,\ v) = \begin{bmatrix} \left\| \underline{r}_c\ (v) - (\underline{r}_c\ (v)\ .\ \underline{d})\ \underline{d} \right\| \cos\ (u + \emptyset) \\ \left\| \underline{r}_c\ (v) - (\underline{r}_c\ (v)\ .\ \underline{d})\ \underline{d} \right\| \sin\ (u + \emptyset) \\ \underline{r}_c\ (v)\ .\ \underline{d} \end{bmatrix} \quad .. \quad (1.11)$$

where

$$\cos \emptyset = \frac{\{\underline{r}_c\ (v) - (\underline{r}_c\ (v)\ .\ \underline{d})\ \underline{d}\}\ .\ \underline{x}}{\left\| \underline{r}_c\ (v) - (\underline{r}_c\ (v)\ .\ \underline{d})\ \underline{d} \right\|}$$

$$0 \leq u \leq 2\pi$$

$\underline{x}$ is the unit vector along the x axis. If the generatrix and axis are coplanar then $\emptyset$ is zero (Then X - Z plane is assumed to be the plane of the generatrix when it is stationary). By substituting appropriate equations for $\underline{r}_c(v)$, surfaces like sphere, prolate ellipsoid, oblate ellipsoid, paraboloid, hyperboloid, annular torus and any surface of general form created by the process of revolution can be represented.

II. <u>Surfaces of general form</u> are obtained when the generating curve moves along a general curved path.

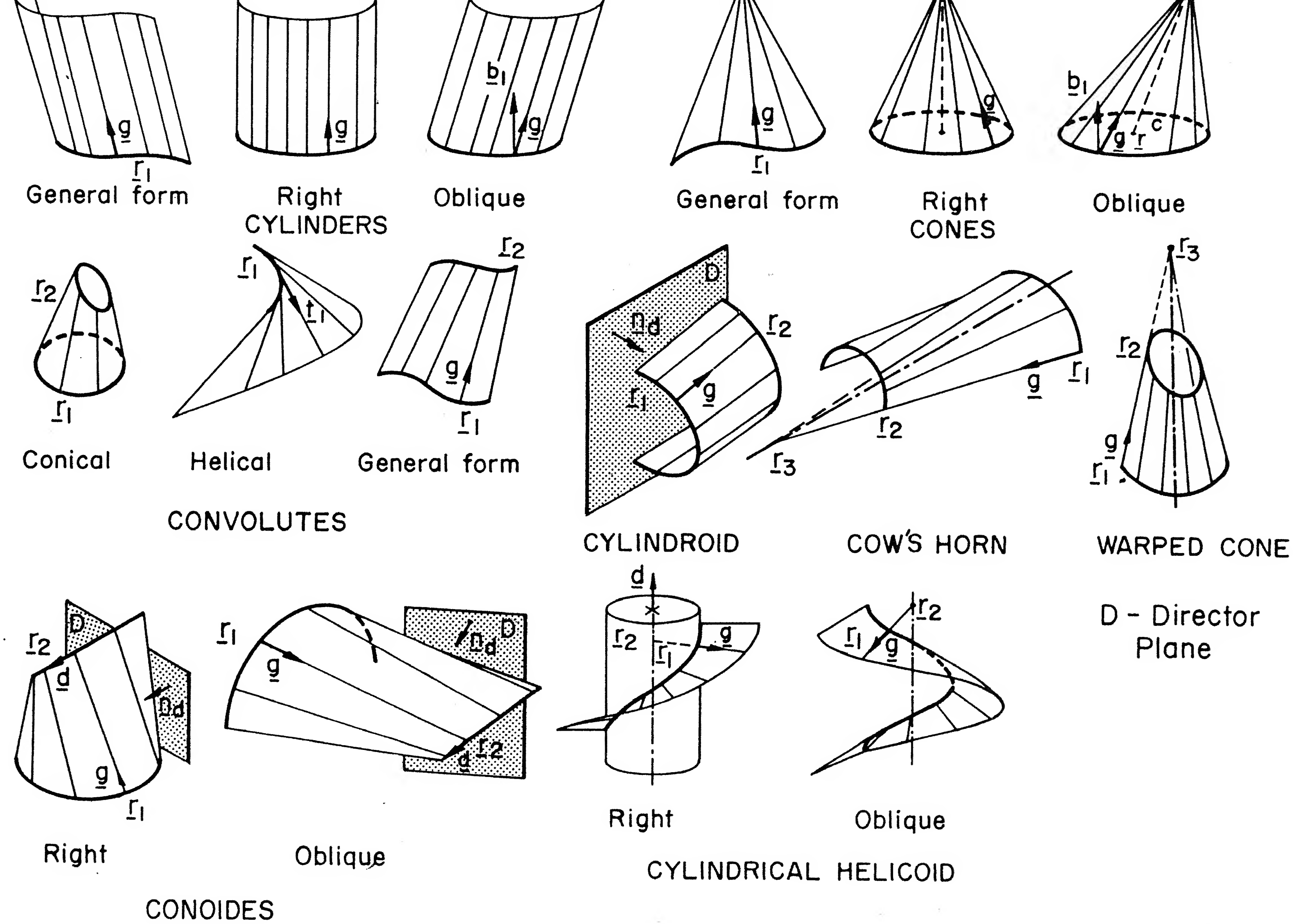


Fig. 1.1 SURFACES.

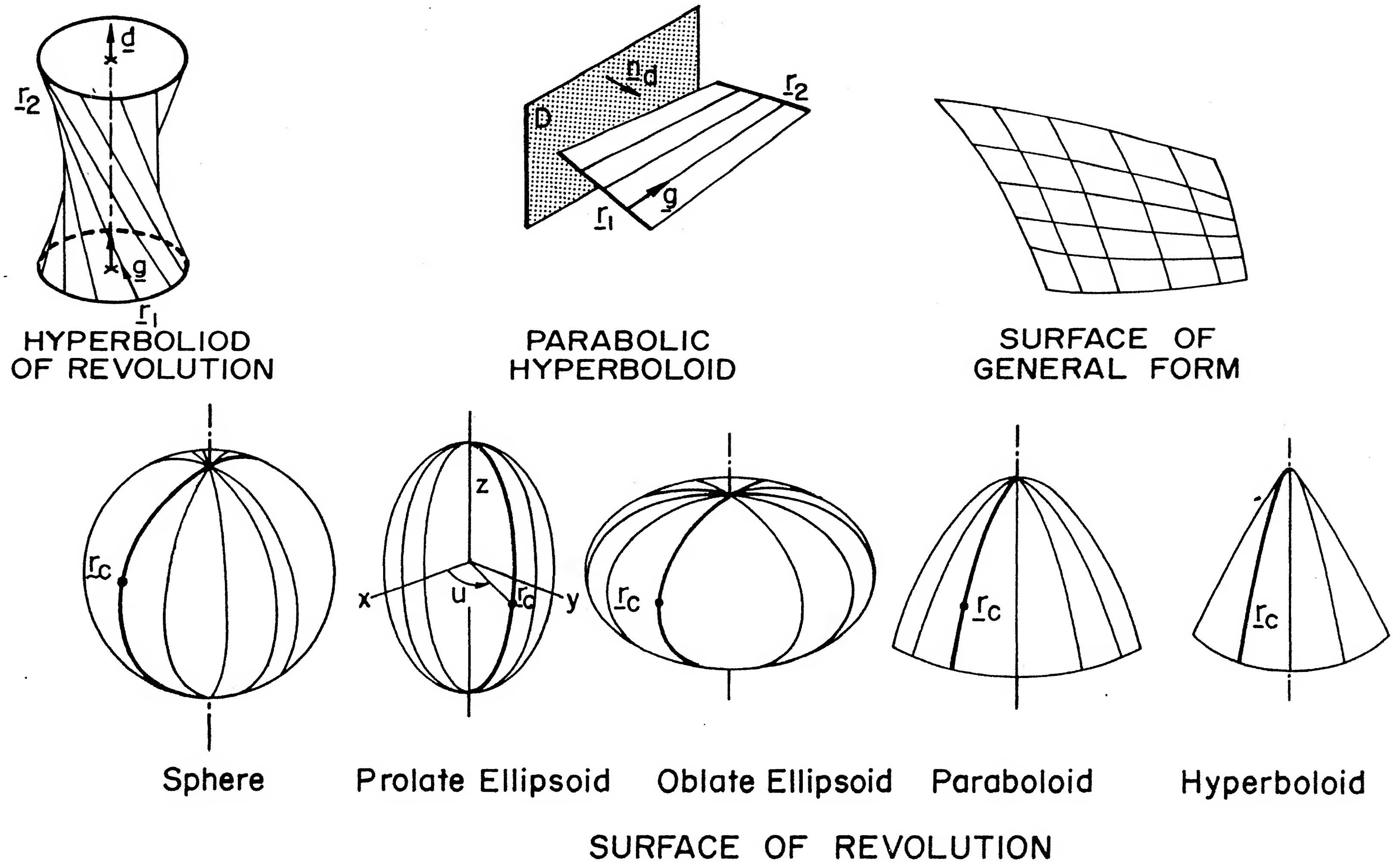


Fig. I.1 SURFACES

1.2.3 Surfaces of General Form

These are free-form or sculptured surfaces. These surfaces are represented by several techniques of computational geometry. Ferguson's cubic surface patch, Bézier's Unisurf surface patches, Coons' patches and many other types of surface patches can be used to define surface of general form [12,13].

1.3 Methods of Development

Depending upon the surface to be developed different methods of development are used. Parallel line method , radial line method and triangulation are essentially the three methods used. Parallel line method is used to develope surfaces of cylinders and prisms wherein the rulings are all parallel to each other. If the rulings form a set of interesting lines as in the case of cones and pyramids, then the radial line method is used. Triangulation method is used for piecewise development of the surface and any single curved ruled surface can be developed by this method. The surface is approximated to a set of triangular planes in contact with each other along the edges. This method is used for finding out the approximate development of non-developable surfaces also.

Graphical methods for developing surfaces by the above methods are described in literature on descriptive geometry and engineering graphics [2-3, 25]. However, these manual methods are time consuming and are subject to drafting inaccuracies. To reduce the time consumed in the development of surfaces and to remove the inaccuracies and dimensional instability associated with manual drafting procedures, the development process can be automated. Suitable mathematical modelling for the development process needs to be developed.

Curves and surfaces have been studied in greater detail and the results are available in literature on analytical geometry and differential geometry. In differential geometry the local properties of curves and surfaces are studied. An area of study in differential geometry is the mapping of one surface onto another surface on a one-to-one correspondence. Different types of mappings such as isometric mapping, isogonal mapping and isoareal mapping had been studied in greater detail and conditions for such mappings had been found by Gauss, Lame, Bonnet, Minding and others [18]. Properties of surfaces and curves associated with the mapping processes had been discussed in detail by Struik [18]. Conditions of developability are mentioned in brief by Struik [18], Faux and Pratt [12].

In literatures on analytical geometry as well as differential geometry only classical surfaces have been studied. Free-form surfaces have been studied in literature on computational geometry. The studies about the development of free-form surfaces are not available in literature.

A mathematical approach to obtain the development of a curved surface is briefly indicated by Faux and Pratt [12]. This approach utilizes the fact that the curvature of the projection of a curve, lying on a surface, onto the tangent plane of the surface (known as Geodesic curvature) and the curvature of the image curve on the development of the surface are equal.

A class of mappings based on an isometric tree has been investigated by Manning [26] and an optimal mapping has been defined for mapping a curved surface onto a plane, with special reference to shoe manufacture. Here also the property of the geodesic curvature is used.

Mathematical modelling and algorithm for the development of transition sections is given by Dhande and Ramulu [27].

An important aspect of the development process is to take into account the thickness of the surface. Classical development procedures described in literature

on descriptive geometry and graphic science consider only thin surfaces. The thickness of the surface is assumed to be negligible. If the surface is thick, this fact is taken care of by giving some suitable allowance. The inner surface of the thick surface is developed by usual methods and the development of the outer surface is obtained by adding some allowance which is determined based on the properties of the material used and the manufacturing process employed to obtain the surface. Thus the development obtained is an approximate one. Studies in the literature on differential geometry and computational geometry also discuss the development of only thin surfaces.

Fournier and Wesley have described a geometric algorithm for performing bending operations on polyhedral objects [28]. At the end of the paper they have discussed the need for a mathematical modelling of the development of thick surfaces.

1.4 Objectives and Scope of the Present Work

If a surface Σ_1 is isometrically mapped onto a planar surface Σ_2 such that the mapping is also isogonal as well as isoareal, then the surface Σ_2 is called the development of the surface Σ_1. Computer aided development of certain classes of ruled surfaces are

considered here. The present study aims at building up relevant mathematical models and suitable algorithms for the development of the following types of surfaces.

(i) Helical convolutes-cylindrical as well as conical,

(ii) Conical convolutes,

(iii) Ducts-planar as well as spatial and

(iv) Thick surfaces-single surface as well as multiple surfaces in series.

Chapter 2

MATHEMATICAL ASPECTS ABOUT DEVELOPMENT OF SURFACES

2.1 Analytic Representation of a Surface

A surface Σ_1 (refer to Figure 2.1a) can be considered to be a set of points in three-dimensional space which is in a one-to-one correspondence with a set of points in a closed rectangle of a plane (refer to Figure 2.1b). The coordinate axes of the rectangular plane are the parameters of the surface Σ_1. Hence the surface Σ_1 is considered to be a bi-parametric surface and a point on it with respect to a coordinate system O-XYZ is given by

$$\underline{r} = \underline{r}\,(u, v) \qquad \begin{array}{l} u_1 \leq u \leq u_2 \\ v_1 \leq v \leq v_2 \end{array} \tag{2.1}$$

where

$$\underline{r}\,(u, v) = \begin{bmatrix} x\,(u, v) \\ y\,(u, v) \\ z\,(u, v) \end{bmatrix} .$$

The bi-parametric surface Σ_1 is generally represented by two families of curves. If the parameter u is kept constant and the other parameter v is varied between v_1 and v_2, then a curve having the value of u constant along

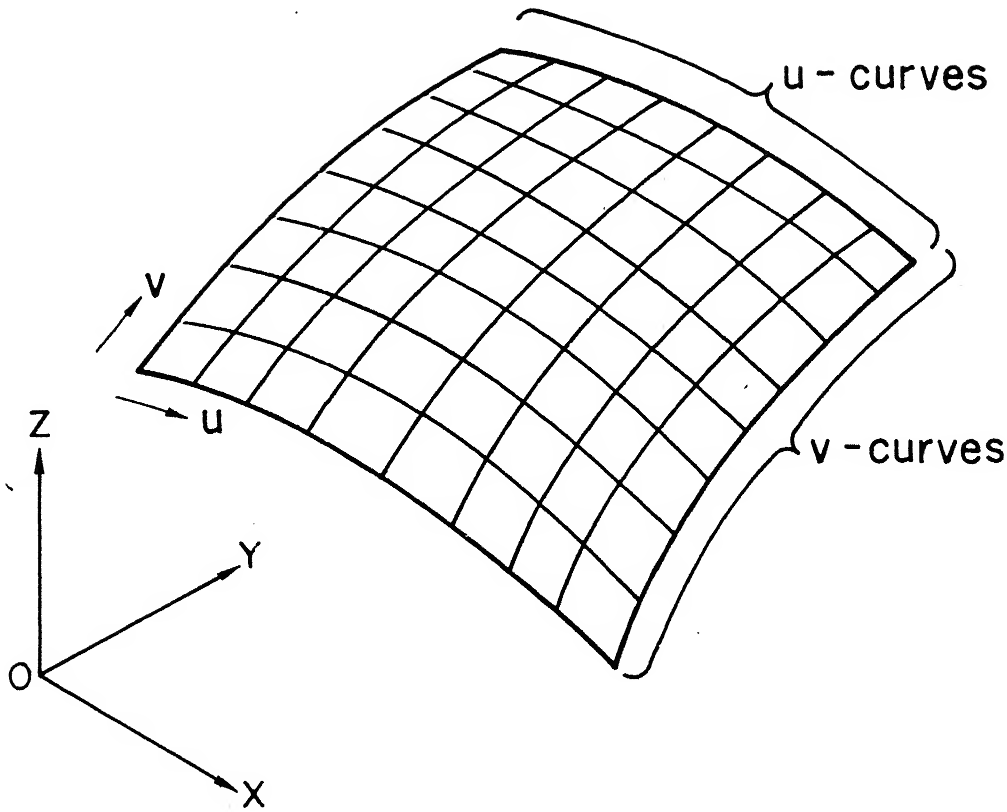


Fig.2.1a Schematic diagram of a bi-parametric surface.

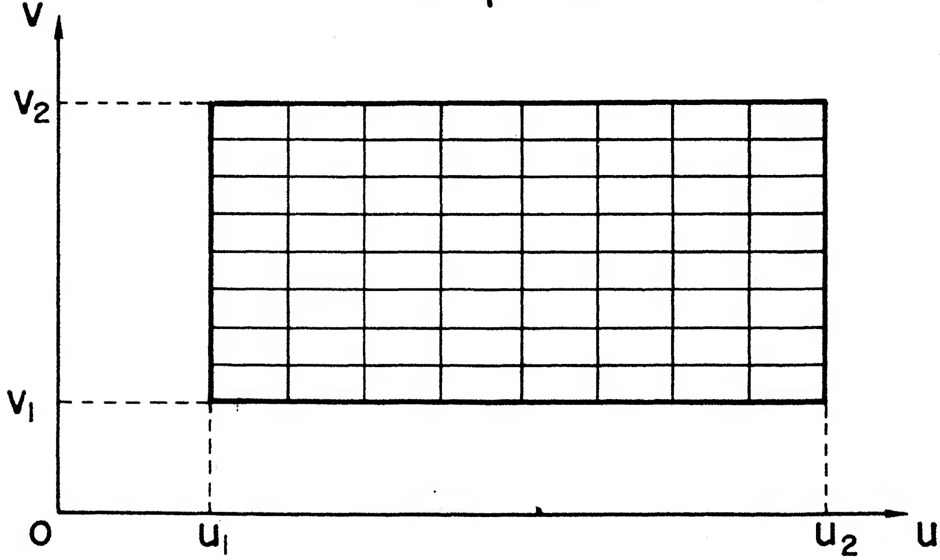


Fig.2.1b The v-u parameter plane.

its length is obtained. A series of such curves called " u-curves" form one family of curves representing the surface. The other family of curves consists of a series of " v-curves" along each one of which the parameter v is kept at some constant value and the other parameter u is varied between u_1 and u_2. At any point P (u, v) on the surface, a pair of one "u-curve" and one "v-curve" pass through it; these curves correspond to the curvilinear parameters u and v of the point P. The generic point P(u, v) on the surface is called a regular point if

$$\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \neq 0. \tag{2.2}$$

This implies that the " u-curve " and the " v-curve " passing through the generic point should have non-zero slope in distinct directions (refer to Figure 2.2). Otherwise the point is called a singular point. In the present work only those surfaces having regular points are considered.

At the generic point P, the tangent plane σ to the surface is defined by the plane containing both the vectors $\frac{\partial \underline{r}}{\partial u}$ and $\frac{\partial \underline{r}}{\partial v}$. The unit normal $\underline{n}_s$ to the surface is given by

$$\underline{n}_s = \frac{\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}}{\left\| \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \right\|} . \tag{2.3}$$

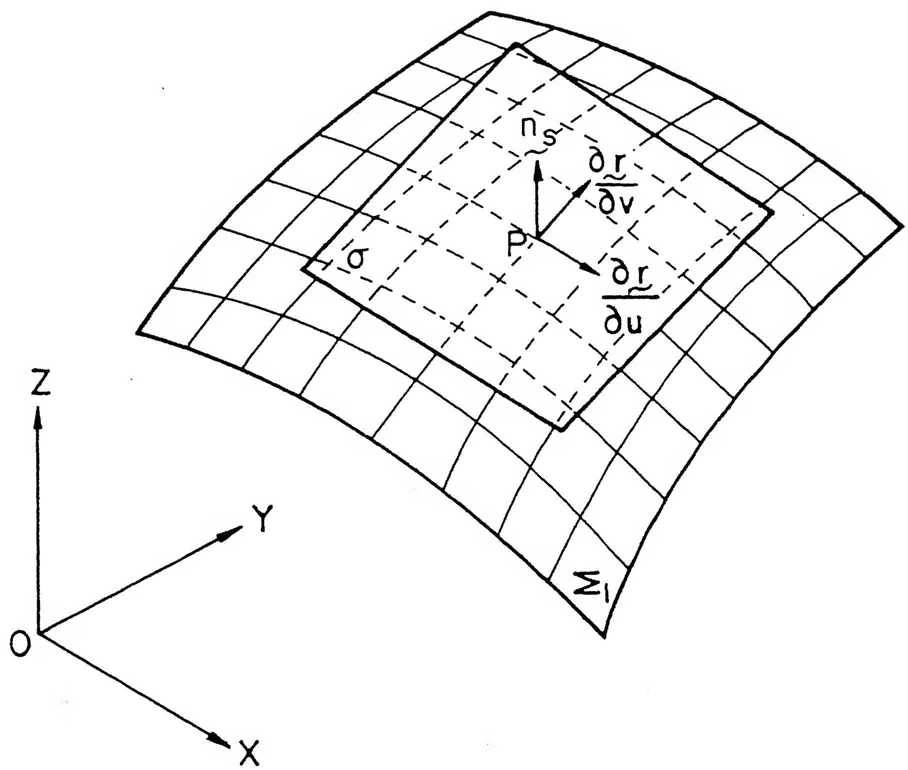


Fig.2.2 The tangent plane.

The parametric equation of the tangent plane is

$$(\underline{r}^{(Q)} - \underline{r}^{(P)}) \cdot \underline{n}_s = 0 \qquad (2.4)$$

where $\underline{r}^{(Q)}$ is the position vector of a generic point Q of the tangent plane σ, $\underline{r}^{(P)}$ is the position vector of the point P on the surface Σ_1 and $\underline{n}_s$ is the unit normal vector at P to the surface Σ_1.

2.2 Developable Surfaces

Consider a one-parameter family of planes and a surface which is an envelope of these planes. The envelope surface and a member plane of the family are tangent to each other along a curve called the characteristic line of the plane. This line is the limiting case of the line of intersection of two infinitesimally separated member planes of the family and, as such, it is a straight line. The envelope is then a surface swept by these characteristic lines which are also called rectilinear generators or rulings of the envelope surface.

A one-parameter family of planes is represented by

$$\underline{n}_s(u) \cdot \underline{r} = \delta (u) \qquad (2.5)$$

where $\underline{n}_s(u)$ is a vector normal to the plane which corresponds to the value of u of the parameter. If the functions $\underline{n}_s(u)$ and $\delta (u)$ are of class C_2 and $\underline{n}_s \cdot (\dot{\underline{n}}_s \times \ddot{\underline{n}}_s) \neq 0$ then the envelope of this family of planes is either a

surface swept by the tangents to some twisted curve in space or a conical surface. If $\underline{n}_s \cdot (\dot{\underline{n}}_s \times \ddot{\underline{n}}_s) = 0$ and the family of planes represented by Eqn. (2.5) does not consist of parallel planes, then the envelope is a cylindrical surface i.e., a surface with all its rulings parallel to each other [17].

In other words, the family of tangent planes of the above mentioned surfaces, such as convolutes, cones and cylinders, is a single-parameter family of planes. Every tangent plane is in contact with the surface along a straight line called the characteristic line of the family of planes. This indicates that the tangent plane at all points along a particular generator coincide. Hence such a surface is developable. It can be developed on the tangent plane.

In short, surfaces which are one-parameter family of planes are called developable surfaces. These surfaces can be mapped isometrically onto a plane. It should be noted that all developable surfaces are ruled surfaces but all ruled surfaces are not developable.

2.3 Ruled Surface, Class-I

Consider a curve C as shown in Figure 2.3. The position vector of a generic point P_1 on the curve is given by $\underline{r}_1(u)$ where u is the parameter of the curve.

Let $\underline{g}(u)$ be a unit vector at point P_1. If $\underline{g}(u)$ defines the direction of a generator of a ruled surface, then the generic point of that ruled surface is given by

$$\underline{r}(u, v) = \underline{r}_1(u) + v\, \underline{g}(u) \tag{2.6}$$

where

$$u_1 \leq u \leq u_2 \quad , \quad v_1 \leq v \leq v_2 .$$

This is a bi-parametric surface ; u and v are the parameters.

2.3.1 Condition for Developability

It is observed in section 2.2 that the tangent planes at all points along a rectilinear generator of a developable surface coincide. Hence the surface normal at all points along a generator is the same. The surface normal for the ruled surface defined by Eqn. (2.6) is given by

$$\underline{N}_s = \dot{\underline{r}}_1(u) \times \underline{g}(u) + v\, \{ \dot{\underline{g}}(u) \times \underline{g}(u) \} \tag{2.7}$$

and

$$\underline{n}_s = \frac{\underline{N}_s}{\| \underline{N}_s \|} .$$

For the surface normal to be the same at all points along the generator, the righthand side of Eqn. (2.7) should be independent of v ; i.e. the vectors $\dot{\underline{r}}_1(u) \times \underline{g}(u)$ and $\dot{\underline{g}}(u) \times \underline{g}(u)$ should be along the same direction. Hence

$$\{ \dot{\underline{r}}_1(u) \times \underline{g}(u) \} \times \{ \dot{\underline{g}}(u) \times \underline{g}(u) \} = 0 \tag{2.8}$$

But $\underline{g}(u)$ is a unit vector and

$$\underline{g} \cdot \underline{g} = 1 \quad ; \quad \dot{\underline{g}} \cdot \underline{g} = 0$$

So Eqn. (2.8) reduces to

$$\underline{g}(u) \cdot \{ \dot{\underline{r}}_1(u) \times \dot{\underline{g}}(u) \} = 0 \tag{2.9}$$

Eqn. (2.9) is the condition for developability.

2.3.2 A Case Study

Two angular parameters are required to define the vector $\underline{g}(u)$ at any point P_1 along the curve C (refer to Figure 2.4). Since only one scalar equation, Eqn. (2.9), has been obtained it is necessary to specify one of the two unknown angular parameters and obtain the other using Eqn. (2.9). The case study presented here illustrates this point.

Let the vector $\underline{g}(u)$ be defined by

$$\underline{g}(u) = \begin{bmatrix} \cos\beta \cos\alpha \\ \cos\beta \sin\alpha \\ \sin\beta \end{bmatrix} \tag{2.10}$$

where α and β are angular parameters and are functions of u (refer to Figure 2.4).

The condition for developability reduces to

$$\frac{d\beta}{du} = \frac{\sin\beta \cos\beta \,(\cos\alpha\, \dot{r}_{1x} + \sin\alpha\, \dot{r}_{1y}) - \cos^2\beta\, \dot{r}_{1z}}{\sin\alpha\, r_{1x} - \cos\alpha\, r_{1y}} \frac{d\alpha}{du} \quad \ldots \tag{2.11}$$

where $\dot{r}_{1x}$, $\dot{r}_{1y}$ and $\dot{r}_{1z}$ are the components of $\dot{\underline{r}}_1$.

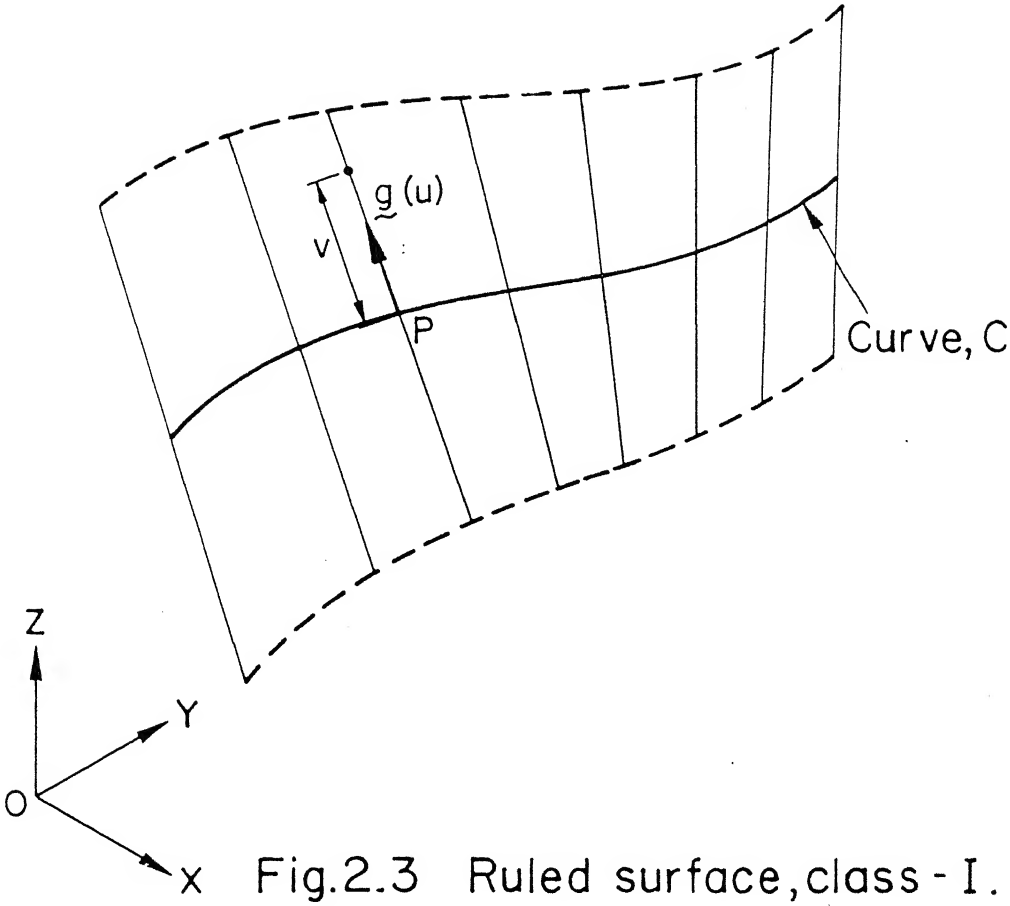


Fig.2.3 Ruled surface, class - I.

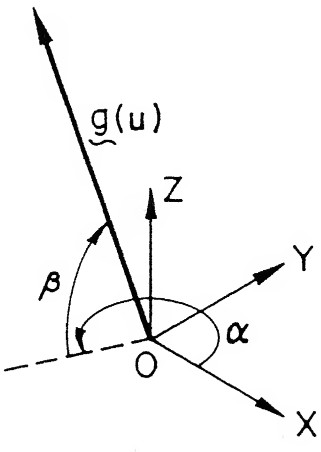


Fig.2.4 Angular parameters of the directional vector.

Given a scalar function $\alpha(u)$, $\frac{d\alpha}{du}$ is a function of u. If the initial value of β is known, then a closed-form analytical solution can be obtained for $\beta(u)$. In general this may not be possible and then a suitable numerical technique, such as the fourth-order Runge-Kutta scheme, can be used to evaluate the value of β at required points along the curve C.

Consider, for example, a curve defined by

$$\underline{r}(u) = \begin{bmatrix} a \cos^{2/n} u \\ b \sin^{2/n} u \\ 0 \end{bmatrix} \tag{2.12}$$

where $n \geq 2$ and $\frac{3\pi}{2} \leq u \leq \frac{7\pi}{2}$. Also consider the angle α to be defined by

$$\alpha = u - \pi. \tag{2.13}$$

Then the Eqn. (2.11), the condition for developability, reduces to

$$\frac{d\beta}{du} = \frac{\sin\beta \cos\beta \sin u \cos u \{ a \sin^{(1-2/n)} u - b \cos^{(1-2/n)} u\}}{a \sin^{(3-2/n)} u + b \cos^{(3-2/n)} u} \quad \ldots\ldots \tag{2.14}$$

If $\beta = \beta_0$ at $u = u_0$ then integration of the above equation yields

$$\tan\beta = \frac{\tan\beta_0 \,[a \sin^{(3-2/n)} u + b \cos^{(3-2/n)} u]^{1/(3-2/n)}}{[a \sin^{(3-2/n)} u_0 + b \cos^{(3-2/n)} u_0]^{1/(3-2/n)}} \quad \ldots\ldots \tag{2.15}$$

Consider two particular cases of the curve,

Case (i) : n = 2, a = b .

Eqns. (2.14) and (2.15) reduce to

$$\frac{d\beta}{du} = 0 \; ; \quad \beta = \text{constant.}$$

The surface considered is a right circular cone.

Case (ii): n = 2 , a $\neq$ b .

Eqns. (2.14) and (2.15) reduce to

$$\frac{d\beta}{du} = \frac{(a-b)\sin\beta\cos\beta\sin u\cos u}{a\sin^2 u + b\cos^2 u}$$

and

$$\tan\beta = \frac{\tan\beta_0}{(a\sin^2 u_0 + b\cos^2 u_0)^{\frac{1}{2}}}\,(a\sin^2 u + b\cos^2 u)^{\frac{1}{2}}$$

For obtaining the values of β using Eqn. (2.11) and given initial condition, a general purpose computer programme has been developed. For the values of n = 2, a = 1.0, b = 0.6, $\beta_0 = \frac{\pi}{3}$ at $u_0 = \frac{3}{2}\pi$, the results obtained using this programme are shown in Figure 2.5.

In conclusion one can observe that a developable ruled surface can be designed with a given curve as its directrix and a straight line as its generatrix by suitably orienting the generatrix as it moves along the directrix.

2.4 Ruled Surface, Class-II

Consider a ruled surface obtained from straight line rulings having a curve C_1 as the primary directrix

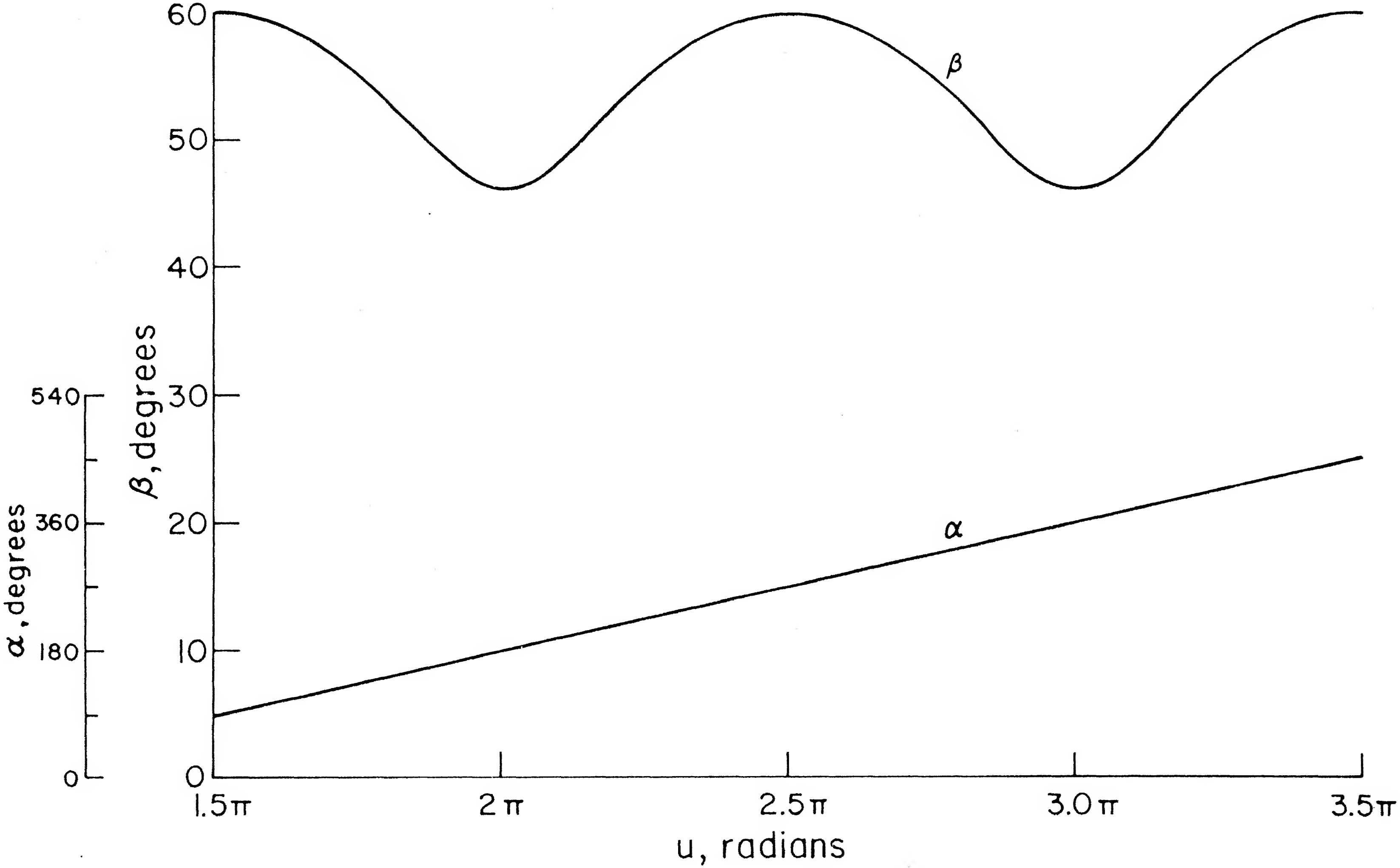


Fig. 2.5 Variation of α and β versus u.

and another curve C_2 as the secondary directrix. A generic point P_1 on C_1 is defined by $\underline{r}_1(u)$, $u_1 \leq u \leq u_2$ and a generic point P_2 on C_2 is defined by $\underline{r}_2(u')$, $u'_1 \leq u' \leq u'_2$. If $P_1 P_2$ is the straight line ruling of the ruled surface, it is required to find the condition for developability of such a surface (refer to Figure 2.6) whose equation is as follows.

$$\underline{r}(u, u', v) = (1 - v)\, \underline{r}_1(u) + v\, \underline{r}_2(u') \qquad (2.16)$$

2.4.1 Condition for Developability

If the surface is developable then the tangent planes at all points along $P_1 P_2$ should coincide. At P_1, the tangent plane passes through $P_1 P_2$ and the tangent to C_1 at P_1. The normal to this tangent plane is given by

$$\underline{N}_1 = \frac{d\underline{r}_1(u)}{du} \times \{ \underline{r}_2(u') - \underline{r}_1(u) \} .$$

Similarly, at P_2, the tangent plane passes through $P_1 P_2$ and the tangent to C_2 at P_2. The normal to this tangent plane is given by

$$\underline{N}_2 = \frac{d\underline{r}_2(u')}{du'} \times \{ \underline{r}_2(u') - \underline{r}_1(u) \} .$$

In order to have the ruled surface to be developable, it is necessary to have these two tangent planes coincide. Hence $\underline{N}_1$ and $\underline{N}_2$ should be parallel. So

$$\underline{N}_1 \times \underline{N}_2 = 0.$$

After substitution and simplification, this condition can be stated as

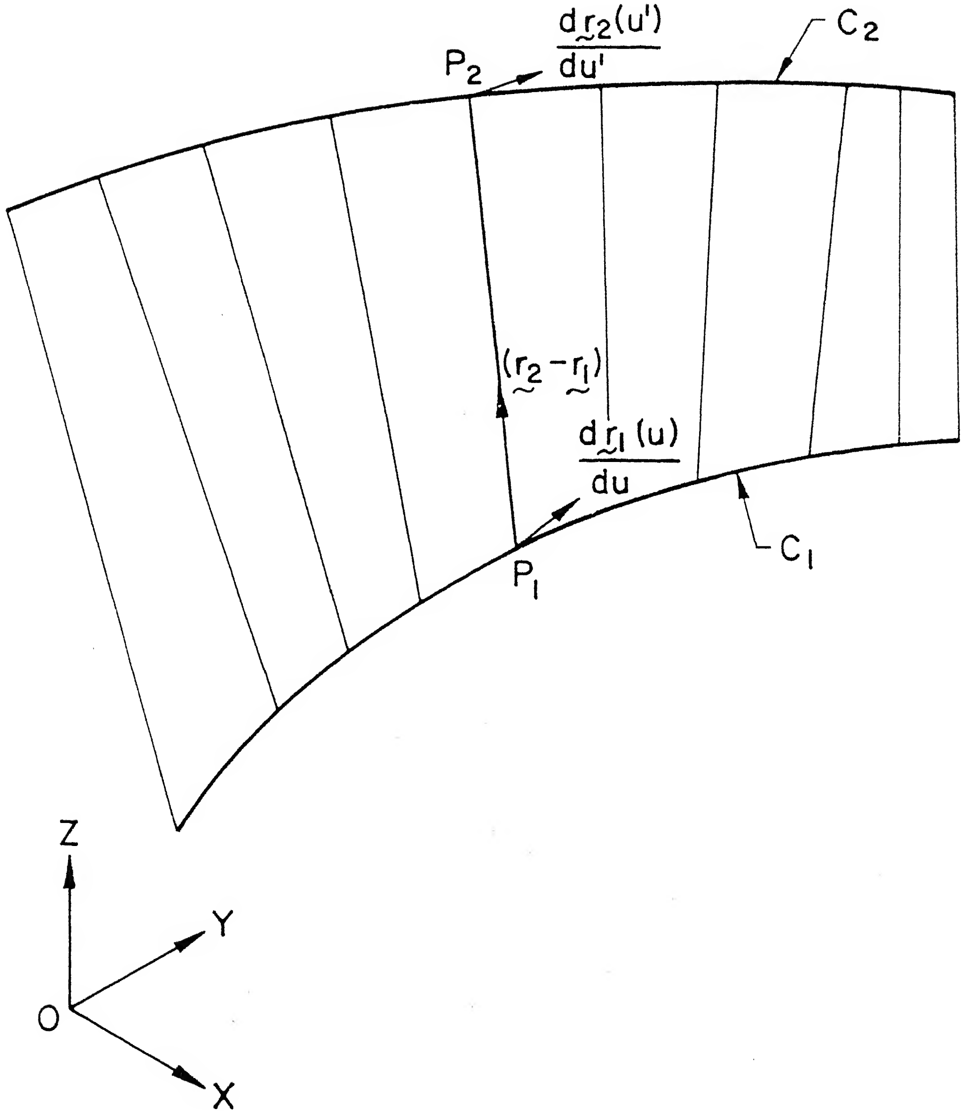


Fig.2.6 Ruled surface, class - II .

$$\{ \underline{r}_2(u') - \underline{r}_1(u) \} \cdot \frac{d\underline{r}_1(u)}{du} \times \frac{d\underline{r}_2(u')}{du'} = 0 \qquad (2.17)$$

Equation (2.17) is the condition for developability of ruled surface, class-II.

2.4.2 Defining a Ruled Surface, Class-II

Equation (2.17) is a scalar equation in two variables u and u'. By fixing a value of one of the variables (say u, i.e. point P_1 on curve C_1), the corresponding value of the other variable (u', i.e. point P_2 on curve C_2) can be fixed. The line joining these two points is the generatrix of the ruled developable surface. For solving Eqn. (2.17), a numerical procedure such as the Bisection Method can be used [29, 30].

Case studies of ruled surfaces, class-II are presented in Chapter 3.

2.5 Mapping

One surface Σ_1 is said to be mapped upon another surface Σ_2 if there is a one-to-one correspondence between their points. Then all points of the two families of curves - u curves and v curves - of surface Σ_1 will be mapped to their corresponding image points on the surface Σ_2 on a one-to-one correspondence resulting in two families of image curves-image u curves and image v curves.

2.5.1 Condition of Mapping

There are three types of mappings. These are (i) isometric mapping, (ii) isogonal mapping and (iii) isoareal mapping. In isometric mapping, the length of an infinitesimal arc on one surface and the length of the corresponding infinitesimal image arc on the other surface are equal. If the angle between two infinitesimal arcs though a point on one surface and the angle between the two infinitesimal image arcs through the corresponding image point on the other surface are equal then the mapping is called isogonal. In isogonal mapping, the length of an infinitesimal arc and of its image are proportional. In isoareal mapping, the area is preserved. The area of an infinitesimal region and of its image are equal.

If the mapping is isometric, it is also isogonal as well as isoareal [17]. Some of the invariants in the case of isometric mapping are the First Fundamental form, the Second Fundamental form, the Gaussian curvature and the Geodesic curvature apart from the length of an arc, the angle between two arcs at a point and the area.

2.5.2 Geodesic Curvature

The curvature of a curve C at a point P on it is given by

$$\underline{K} = k\,\underline{n}_c \qquad (2.18)$$

where $\underline{n}_c$ is the principal normal to C at P.

If the curve C lies on a surface Σ , this curvature vector can be decomposed into two components - one normal to the surface and the other tangential to the surface. The normal component is the normal curvature $\underline{K}_n$ and the tangential component is the geodesic curvature $\underline{K}_g$ (refer to Figure 2.7). This component is taken along the direction of a unit vector $\underline{i}_1$ in the tangent plane to the surface such that the tangent vector to the curve and this unit vector $\underline{i}_1$ are at right angles and in the sense of X and Y in a right handed co-ordinate system.

If $\underline{b}$ is the binormal vector to C at P and $\underline{r}_c$ is the radius vector of P as a point on the curve C then

$$k\,\underline{b} = \frac{\dot{\underline{r}}_c \times \ddot{\underline{r}}_c}{\dot{s}_c^{\,3}} \tag{2.19}$$

where $\quad \underline{r}_c = \underline{r}_c\,(t),\ \dot{\underline{r}}_c = \dfrac{d\,\underline{r}_c}{dt},\ \ddot{\underline{r}}_c = \dfrac{d^2\underline{r}_c}{dt^2},\ \dot{s}_c = \|\dot{\underline{r}}_c\|$.

The angle between the vectors $\underline{b}$ and $\underline{n}_s$ (unit normal to the surface) is the same as that between the vector $\underline{n}_c$ and $\underline{i}_1$. Hence the magnitude of the geodesic curvature k_g is given by

$$k_g = \underline{n}_s \cdot \frac{\dot{\underline{r}}_c \times \ddot{\underline{r}}_c}{\dot{s}_c^{\,3}} \tag{2.20}$$

Geodesic curvature is a bending invariant. In an isometric mapping, the geodesic curvature value is preserved. The geodesic curvature of an arc on a surface

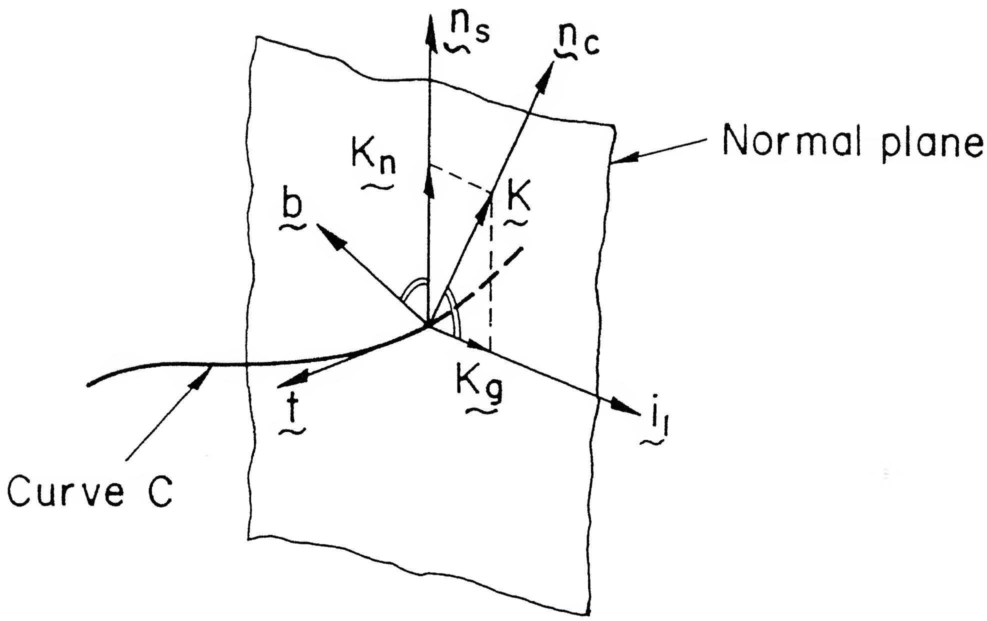


Fig. 2.7 Geodesic curvature.

Σ_1 and that of its isometrically mapped image arc on surface Σ_2 are equal. If the surface Σ_2 is a planar surface then the image arc is a planar curve and its geodesic curvature is nothing but its planar curvature.

2.6 Algorithm for Development of Surfaces

The isometric mapping of a ruled surface Σ_1 onto a planar surface Σ_2 lying along the plane tangent to the straight line generator of the surface Σ_1 is called the development of the surface Σ_1. The length of an infinitesimal arc, the angle between two infinitesimal arcs at a point, the area and the geodesic curvature value are all preserved in the development image.

Based on the foregoing concepts of isometric mapping and geodesic curvature, two algorithms for development of ruled surfaces of class-I and class-II are given below.

2.6.1 Ruled Surface, Class-I

Step 1. The surface is defined using Eqn. (2.6). The directrix is defined completely.

Step 2. The length of the straight line generatrix is fixed.

Step 3. The direction of the generatrix is found based on the condition for developability using Eqn. (2.9). If the direction is that of the tangent to

the directrix, the surface is called a tangent developable surface.

Step 4. Corresponding to the various positions of the generic point along the directrix, the following quantities are evaluated.

(a) geodesic curvature,

(b) length of arc from the starting point,

(c) arc-tangent angle. This is the angle made by the tangent to the directrix at a generic point with respect to the tangent to the directrix at the starting point.

(d) angle between the arc-tangent and the generatrix.

Step 5. The directrix is isometrically mapped. This is carried out by integrating Serret-Frenet equations [12] (refer to Figure 2.8)

$$\frac{d^2x}{ds^2} + k_g(s)\frac{dy}{ds} = 0$$
$$\frac{d^2y}{ds^2} - k_g(s)\frac{dx}{ds} = 0 \qquad (2.21)$$

where

$k_g(s)$ - geodesic curvature

x,y - coordinates of the developed curve in an O-XY plane where O is the starting point of the mapped curve and the X-axis is along the direction of the tangent vector at the starting point

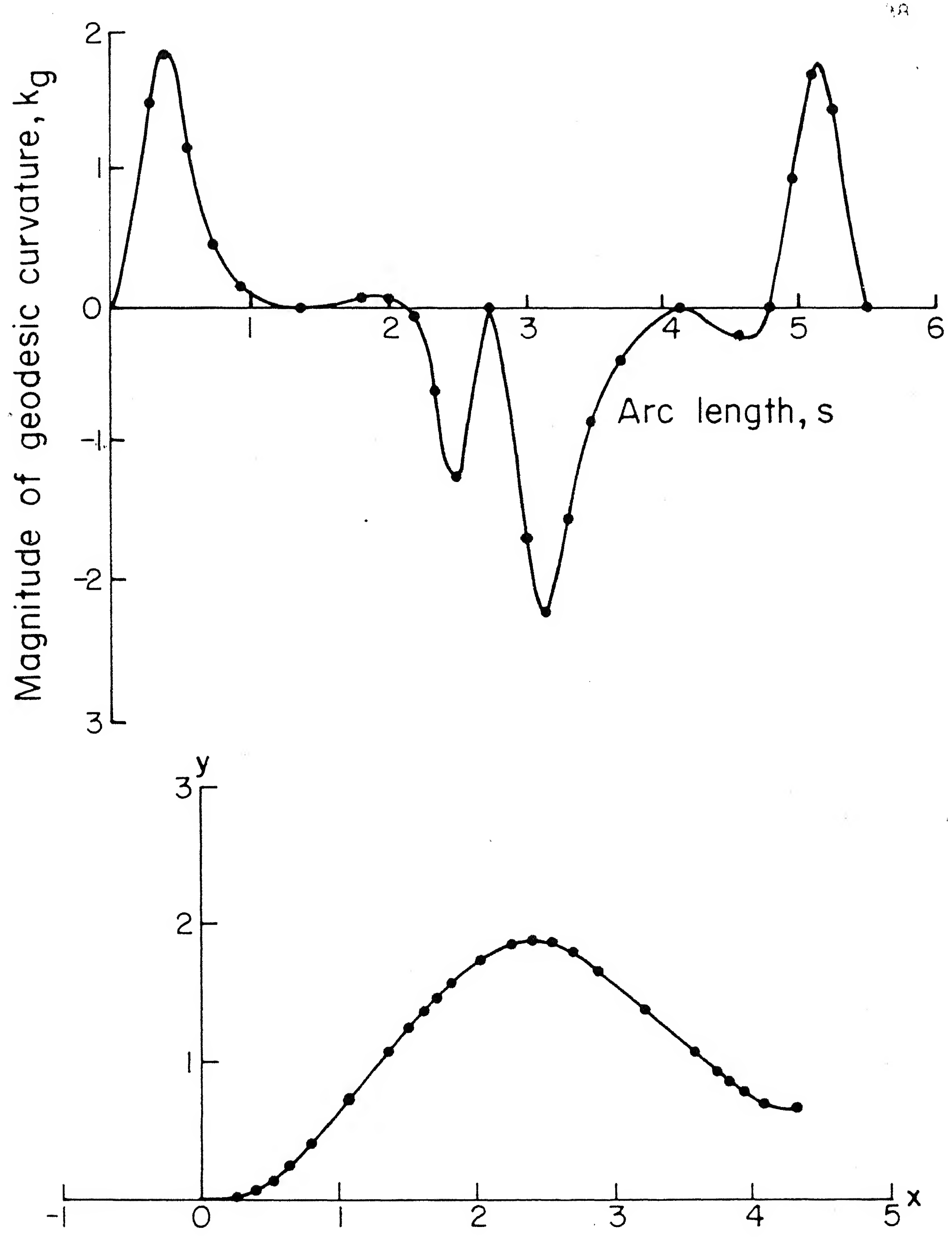


Fig. 2.8 Graphs of magnitude of geodesic curvature vs. arc length and the x-y development.

s - length of arc from the starting point.

Step 6. The generatrix is mapped. The angle between the arc and the generatrix as well as the length of the generatrix are maintained. A curve is drawn passing through the other end of the generatrix at its various positions.

This completes the development of the surface.

2.6.2 Ruled Surface, Class-II

Step 1. The surface is completely defined as in Eqn. (2.16). The directrices are completely defined.

Step 2. Corresponding to various positions of a generic point on the primary directrix, the position of the suitable point on the secondary directrix is found out so as to satisfy the condition for developability using Eqn. (2.17).

Step 3. Corresponding to the various positions of the generic point on the primary directrix the following quantities are evaluated.

(a) length of the generatrix

(b) geodesic curvature

(c) length of arc (primary directrix) from the starting point

(d) arc-tangent angle for the primary directrix

(e) angle between the arc-tangent and the generatrix.

Step 4. The primary directrix is isometrically mapped by carrying out the integration of Serret-Frenet Equations (as mentioned in Step 5 for Ruled Surface, Class-I).

Step 5. The generatrix is mapped with respect to the primary directrix isogonally. The curve passing through various positions of the other end of the generatrix gives the mapping of the secondary directrix.

The development of the surface is thus obtained.

Chapter 3

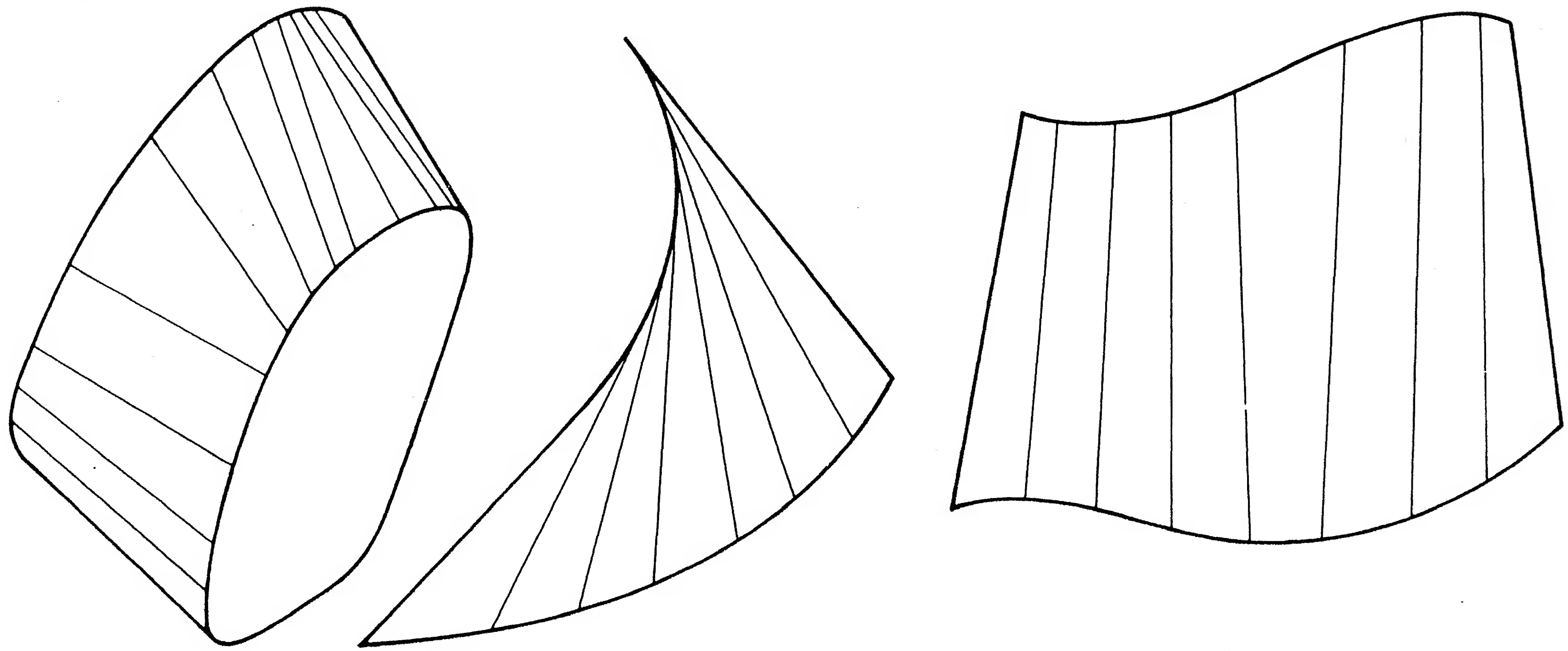
DEVELOPMENT OF CONICAL CONVOLUTES

A convolute is a surface generated by a moving plane tangent to two space curves; the surface is the locus of the elements of tangency [4]. There are two forms; conical and helical convolutes. A conical convolute, a helical convolute and a convolute of general form are shown in Figure 3.1. The development of conical convolute surface is dealt with in this chapter and the development of helical convolute is treated in the next chapter.

3.1 Conical Convolute

Conical convolute is also commonly known as conical transition. It is generated by a straight line generatrix moving in contact with two directrices such that the generatrix always lies on a plane tangent to the two directrices. The adjacent elements lie infinitely close together in a plane and will be intersecting [2].

If the directrices are planar ellipses, these can be represented parametrically by



Conical convolute   Helical convolute   Convolute, General form

Fig. 3.1 Convolutes.

$$\underline{r} = \begin{bmatrix} a \cos \theta \\ b \sin \theta \\ 0 \\ 1 \end{bmatrix} \qquad 0 \leq \theta \leq 2\pi \tag{3.1}$$

where $\underline{r}$ is the position vector of a generic point on the directrix with semi-major diameter equal to a and semi-minor diameter equal to b. If a = b, then the directrix is a circle.

3.2 Super-Conical Convolute

If atleast one of the directrices is a super-ellipse then the convolute can be called as super-conical convolute. The parametric representation of a super-ellipse is given by

$$\underline{r} = \begin{bmatrix} a \cos^{2/n} \theta \\ b \sin^{2/n} \theta \\ 0 \\ 1 \end{bmatrix} \tag{3.2}$$

where

a - semi-major diameter of the super-ellipse,

b = semi-minor diameter of the super-ellipse,

n = power index for the super-ellipse (n > 0).

For n = 2, Eqn. (3.2) represents an ordinary ellipse (Eqn. (3.1)). For n > 2, the fullness of the ellipse increases with increasing value of n. The curve

approximates to a rectangle when $n \rightarrow \infty$ [12,13] (refer to Figure 3.2). For $n < 2$ and as $n \rightarrow 1$, the super-ellipse approximate to a rhombus. For $n = 1$, it is a rhombus. Further as n decreases below 1, the super-ellipse takes a shape shown by curve A in Figure 3.2. It can be seen that a variety of shapes can be obtained by the same parametric equation (Eqn. (3.2)) by changing the value of index n.

3.3 Super-Conical Convolute: Spatial Configuration

One of the directrices of the convolute is called the primary directrix and the other the secondary. Suffices i and j are used with the primary and the secondary directrices respectively. The spatial configuration of the super-conical convolute is defined by specifying with respect to a global co-ordinate frame

(a) the orientation of the planes of the directrices,
(b) the location of the centres of the super-ellipses and
(c) the direction of the two axes of the super-ellipses.

This has been done in two ways as described below. In both the cases the local co-ordinate frames for the directrices are chosen to be right-handed one with X and Y axes lying respectively along the major and minor diameters

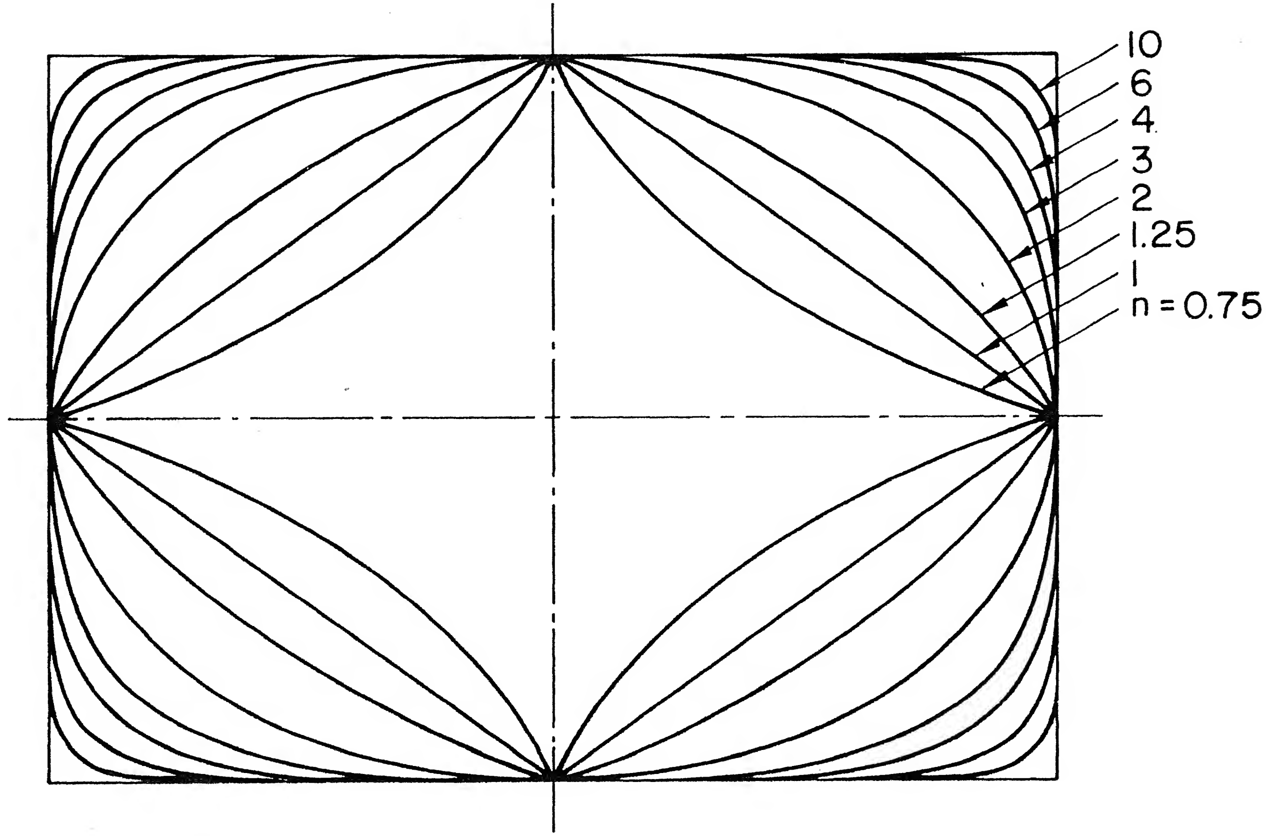


Fig. 3.2 Super - ellipses .

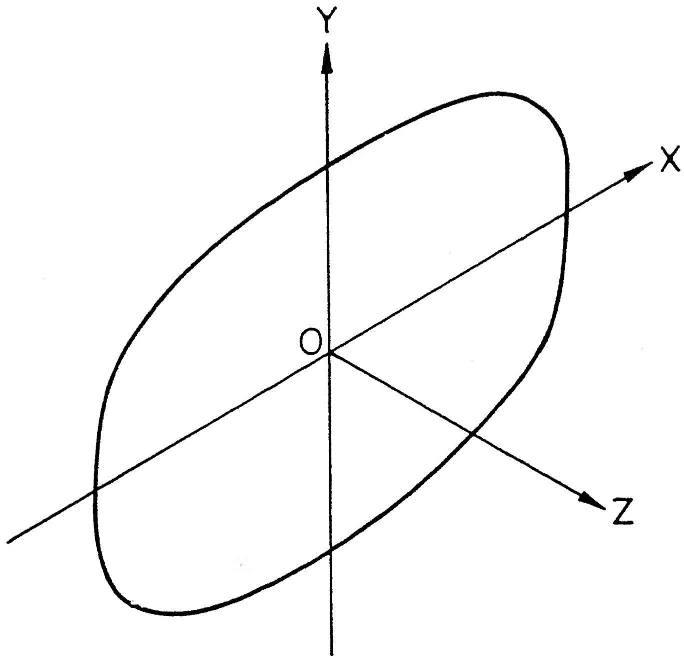
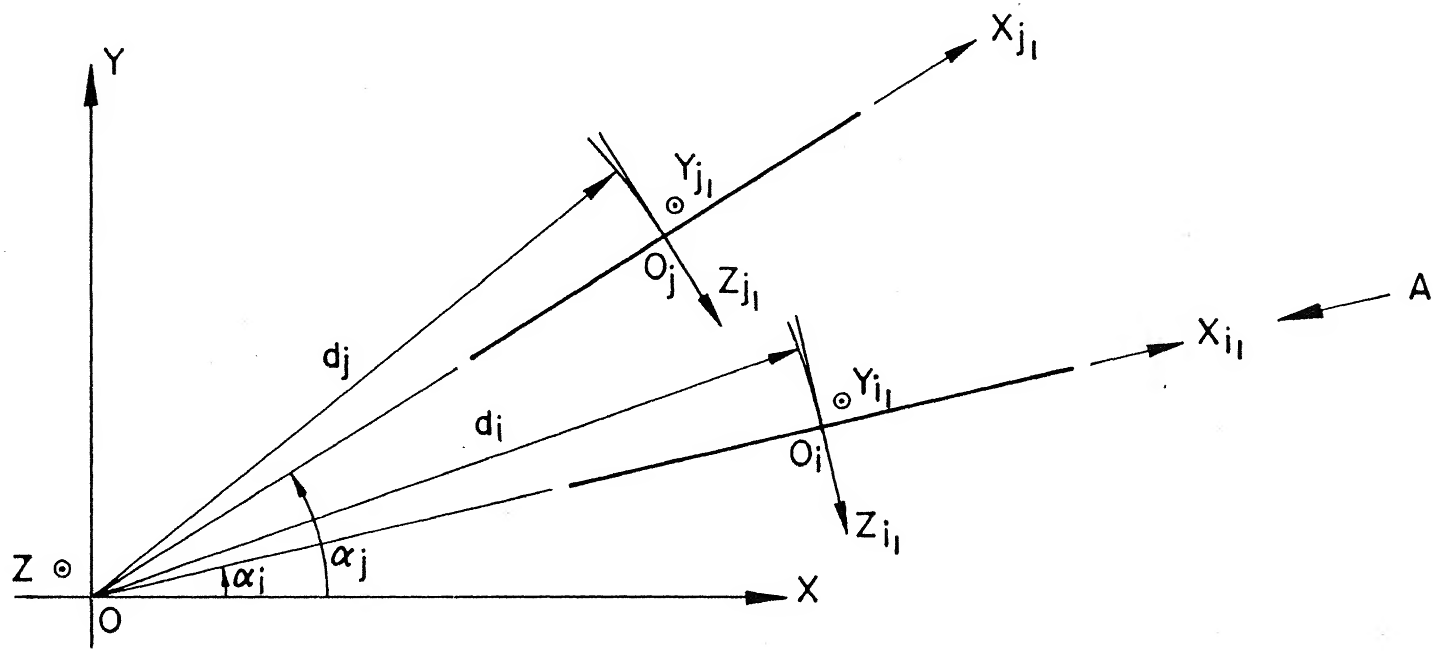


Fig. 3.3 Super-ellipse and local co-ordinate frame.

of the super-ellipses and the z axis lying along the normal to the planes of the super-ellipses (refer to Figure 3.3).

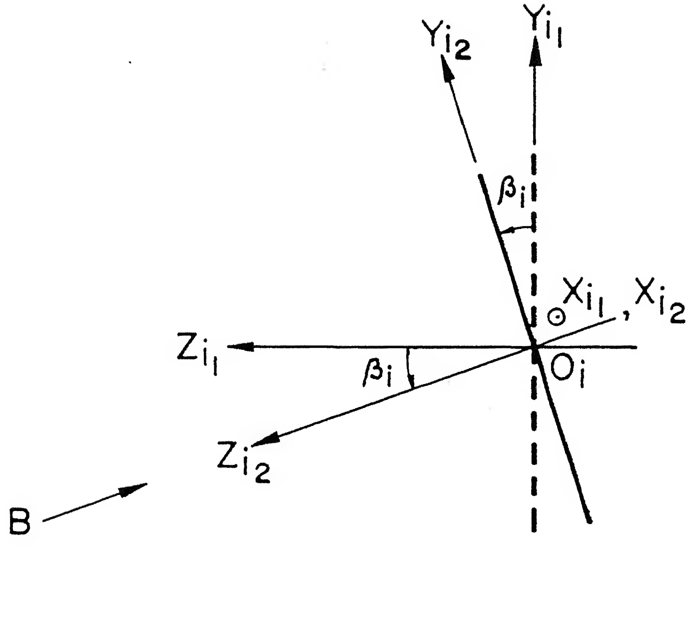
3.3.1 Spatial Configuration-Definition (A)

A global co-ordinate frame O-XYZ is defined such that the centres O_i and O_j of the two super-ellipses lie on the O-XY plane and the lines of intersection of the planes of the super-ellipses with the O-XY plane meet at O and are at angles α_i and α_j to the x axis (Figure 3.4). The centres of the two super-ellipses are at a distance d_i and d_j respectively from the origin O. The orientation of the planes of the super-ellipses and their axes is defined to be obtained by the method described here. The planes of the super-ellipses are first assumed to be perpendicular to the O-XY plane, with the major diameters of the super-ellipses lying on the O-XY plane (Figure 3.4a). The planes of the super-ellipses are first rotated about the respective major diameters through angles β_i and β_j. The curves are then rotated through angles γ_i and γ_j about the normal to their planes through their centres O_i and O_j respectively. This completes the definition of the spatial configuration of the super-conical convolute. In Figure 3.4b rotation of the plane of primary directrix through an angle β_i about its major diametre is shown. Figure 3.4c shows the rotation of the primary directrix through an angle γ_i about the normal to

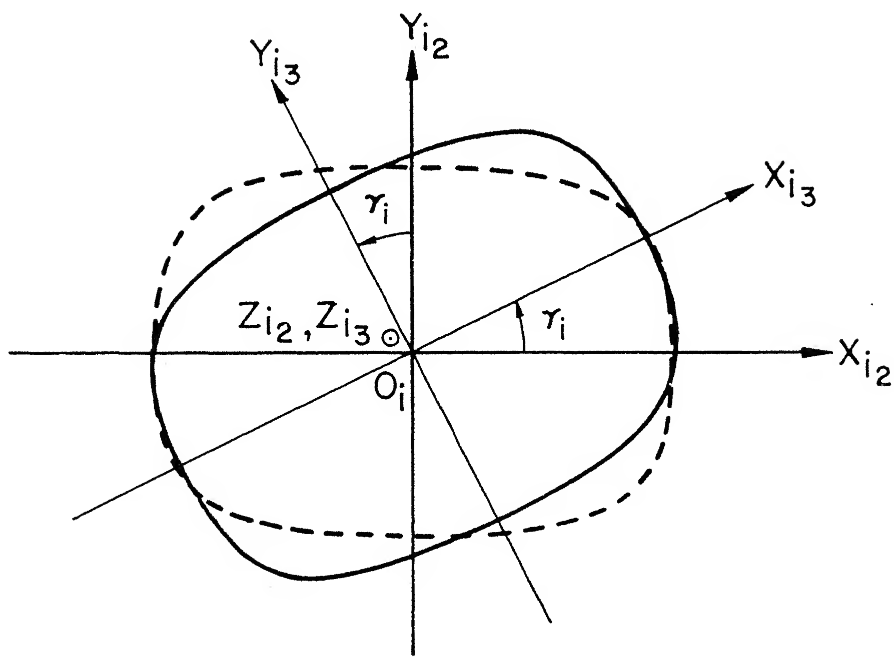


(a)

Fig.3.4 Super-conical convolute. Spatial configuration : Definition (A)



(b) View A



(c) View B

Fig.3.4 Super-conical convolute.
Spatial configuration: Definition (A)

its plane through O_i . $O_i-X_{i_3}Y_{i_3}Z_{i_3}$ is the final local co-ordinate frame for the primary directrix. Figures similar to 3.4b and 3.4c can be drawn for the secondary directrix.

3.3.2 Spatial Configuration-Definition (B)

Here the centres O_i and O_j of the two super-ellipses are chosen to be two points along a space curve defined in the global co-ordinate frame. The planes of the super-ellipses are taken to be the normal planes of the space curve at points O_i and O_j. First the major and minor diameters of the super-ellipses are aligned along the normal and bi-normal to the space curve at O_i and O_j. Of course the normals to the planes of the super-ellipses are along the tangents to the space curve at O_i and O_j. Then the super-ellipses are rotated about these normals through angles γ_i and γ_j respectively. This gives the final configuration (refer to Figure 3.5).

3.3.3 Transformation Matrix

Consider, for example, the super-ellipse acting as the primary directrix. The position vector of a generic point of the super-ellipse is given by

$$\underline{r}_i = [T_i]\, \underline{r}_{L_i} \tag{3.3}$$

where

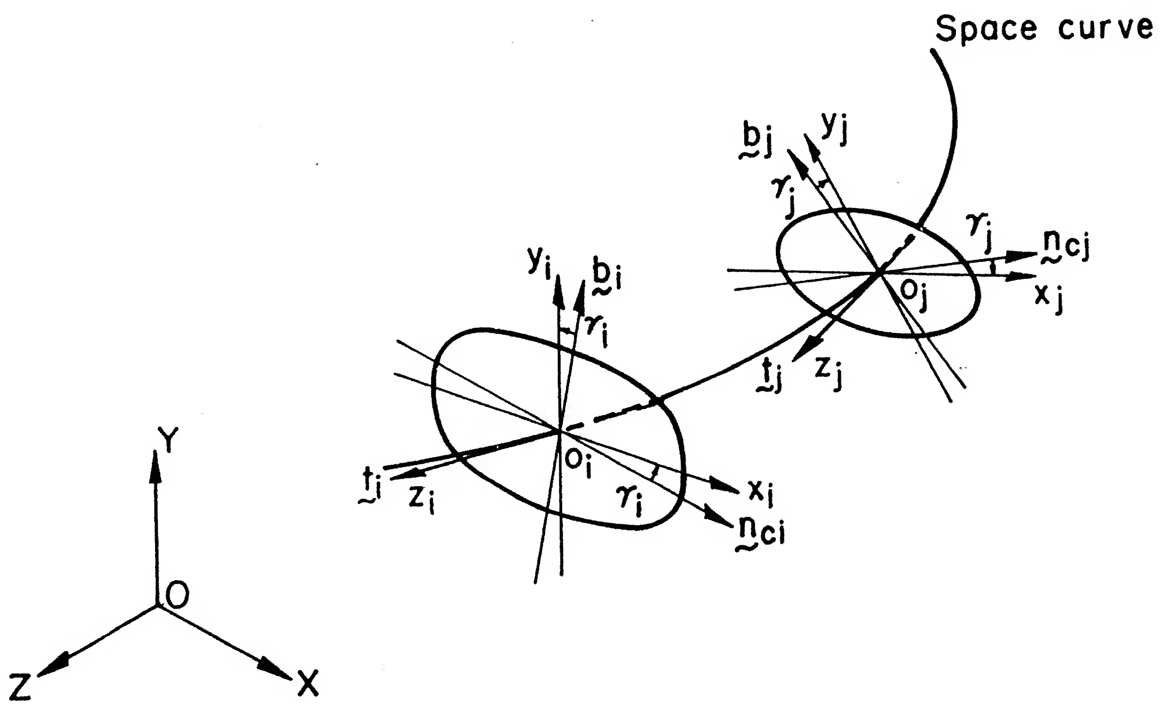


Fig.3.5 Super-conical convolute. Spatial configuration: Definition (B)

$\underline{r}_i$ - the position vector of the generic point in global co-ordinates,

$\underline{r}_{L_i}$ - the position vector of the generic point in local co-ordinates and

$[T_i]$ - the transformation matrix to convert values in local co-ordinates to those in global co-ordinates.

In homogeneous co-ordinate system, Eqn. (3.3) is expressed as

$$\begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix} = [T_i] \begin{bmatrix} a_i \cos^{2/n_i} \theta_i \\ b_i \sin^{2/n_i} \theta_i \\ 0 \\ 1 \end{bmatrix} \qquad (3.4)$$

where $[T_i]$ is a 4 x 4 matrix. Values of its elements depend upon the way the spatial configuration of the super-ellipse is defined.

As per definition (A) the transformation matrix $[T_i]$ is obtained as

$$[T_i] = \begin{bmatrix} t_{i_{1,1}} & t_{i_{1,2}} & t_{i_{1,3}} & t_{i_{1,4}} \\ t_{i_{2,1}} & t_{i_{2,2}} & t_{i_{2,3}} & t_{i_{2,4}} \\ t_{i_{3,1}} & t_{i_{3,2}} & t_{i_{3,3}} & t_{i_{3,4}} \\ t_{i_{4,1}} & t_{i_{4,2}} & t_{i_{4,3}} & t_{i_{4,4}} \end{bmatrix} \qquad (3.5)$$

where

$$t_{i_{1,1}} = \cos\alpha_i \cos\gamma_i + \sin\alpha_i \sin\beta_i \sin\gamma_i$$

$$t_{i_{1,2}} = -\cos\alpha_i \sin\gamma_i + \sin\alpha_i \sin\beta_i \cos\gamma_i$$

$$t_{i_{1,3}} = \sin\alpha_i \cos\beta_i$$

$$t_{i_{1,4}} = d_i \cos\alpha_i$$

$$t_{i_{2,1}} = \sin\alpha_i \cos\gamma_i - \cos\alpha_i \sin\beta_i \sin\gamma_i$$

$$t_{i_{2,2}} = -\sin\alpha_i \sin\gamma_i - \cos\alpha_i \sin\beta_i \cos\gamma_i$$

$$t_{i_{2,3}} = -\cos\alpha_i \cos\beta_i$$

$$t_{i_{2,4}} = d_i \sin\alpha_i$$

$$t_{i_{3,1}} = \cos\beta_i \sin\gamma_i$$

$$t_{i_{3,2}} = \cos\beta_i \cos\gamma_i$$

$$t_{i_{3,3}} = -\sin\beta_i$$

$$t_{i_{3,4}} = 0$$

$$t_{i_{4,1}} = 0$$

$$t_{i_{4,2}} = 0$$

$$t_{i_{4,3}} = 0$$

$$t_{i_{4,4}} = 1$$

Similarly, definition (B) yields the transformation matrix

$$[T_i]=\begin{bmatrix} n_{c_{i_x}} & b_{i_x} & t_{i_x} & r_{o_{i_x}} \\ n_{c_{i_y}} & b_{i_y} & t_{i_y} & r_{o_{i_y}} \\ n_{c_{i_z}} & b_{i_z} & t_{i_z} & r_{o_{i_z}} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\gamma_i & -\sin\gamma_i & 0 & 0 \\ \sin\gamma_i & \cos\gamma_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.6)$$

where $\underline{n}_{c_i}$, $\underline{b}_i$ and $\underline{t}_i$ are the unit normal, bi-normal and tangent vectors at O_i to the space curve; $\underline{r}_{o_i}$ is the position vector of point O_i. The subscripts x, y and z indicate the component of these vectors with respect to the global co-ordinate frame. The transformation matrix to be used for the secondary directrix can be obtained by replacing the subscript i by j in Eqns. (3.5) and (3.6).

3.4 Development

Given the details about the two super-ellipses acting as directrices and the spatial configuration of the super-conical convolute, the development of the super-conical convolute is carried out following the algorithm given in section 2.6. First the transformation matrices for the directrices are found out. Then the various quantities required for the development of the super-conical convolute are calculated. The mathematical expressions for them are given below.

3.4.1 Generic Point of a Super-Ellipse

The parametric representation of a generic point of the super-ellipse treated as the primary directrix is given, in global co-ordinates, by

$$\underline{r}_i = [T_i] \begin{bmatrix} a_i \cos^{2/n_i} \theta_i \\ b_i \sin^{2/n_i} \theta_i \\ 0 \\ 1 \end{bmatrix} \qquad (3.7)$$

where $[T_i]$ is the transformation matrix. The tangent vector to the primary directrix at the generic point is given by

$$\dot{\underline{r}}_i = \frac{d\underline{r}_i}{d\theta_i}$$

$$= \frac{2}{n_i} \cos^{(2/n_i - 1)} \theta_i \sin^{(2/n_i - 1)} \theta_i \, [T_i^*] \begin{bmatrix} -a_i \sin^{(2-2/n_i)} \theta_i \\ b_i \cos^{(2-2/n_i)} \theta_i \\ 0 \end{bmatrix} \quad \ldots \quad (3.8)$$

where $[T_i^*]$ is the 3 x 3 matrix obtained by deleting the last row and column of the matrix $[T_i]$. The magnitude of the tangent vector is given by

$$\|\dot{\underline{r}}_i\| = \dot{s}_i = \frac{2}{n_i} \cos^{(2/n_i - 1)} \theta_i \sin^{(2/n_i - 1)} \theta_i$$

$$\left[\sum_{k=1}^{3} \{ -t_{i_{k,1}} a_i \sin^{(2-2/n_i)} \theta_i + t_{i_{k,2}} b_i \cos^{(2-2/n_i)} \theta_i \}^2 \right]^{1/2} \qquad (3.9)$$

where $t_{i_{k,1}}$, for example, is the element of the matrix $[T_i]$ in the k^{th} row and 1^{th} column. The unit tangent vector at the generic point is given by

$$\underline{t}_i = \frac{[T_i^*]}{(\text{Factor } 1)_i} \begin{bmatrix} -a_i \sin^{(2-2/n_i)}\theta_i \\ b_i \cos^{(2-2/n_i)}\theta_i \\ 0 \end{bmatrix} \qquad (3.10)$$

where

$$(\text{Factor } 1)_i = \left[\sum_{k=1}^{3} \{ -t_{i_{k,1}} a_i \sin^{(2-2/n_i)}\theta_i + t_{i_{k,2}} b_i \cos^{(2-2/n_i)}\theta_i \}^2 \right]^{\frac{1}{2}} \qquad .. \quad (3.11)$$

The second derivative to the vector $\underline{r}_i$ is given by

$$\ddot{\underline{r}}_i = \frac{d^2\underline{r}_i}{d\theta_i^{\,2}}$$

$$= \frac{2}{n_i} \cos^{(2/n_i-2)}\theta_i \sin^{(2/n_i-2)}\theta_i [T_i^*] \begin{bmatrix} (2/n_i \sin^2\theta_i - 1)a_i \sin^{(2-2/n_i)}\theta_i \\ (2/n_i \cos^2\theta_i - 1)b_i \cos^{(2-2/n_i)}\theta_i \\ 0 \end{bmatrix} \qquad .. \quad (3.12)$$

By substituting j for i in Eqns. (3.7) to (3.12) the corresponding quantities for the super-ellipse treated as secondary directrix are obtained.

3.4.2 Condition for Developability

As has already been mentioned in Chapter 2 (Eqn. (2.17))the condition for developability (of the super-convolute) is given by

$$(\underline{r}_i - \underline{r}_j) \cdot (\dot{\underline{r}}_i \times \dot{\underline{r}}_j) = 0 \ .$$

Substituting for $\underline{r}_i$, $\underline{r}_j$, $\dot{\underline{r}}_i$ and $\dot{\underline{r}}_j$ and simplifying, the condition for developability reduces to

$$(2/n_i)(2/n_j)\cos^{(2/n_i-1)}\theta_i \ \sin^{(2/n_i-1)}\theta_i \ \times$$

$$\cos^{(2/n_j-1)}\theta_j \ \sin^{(2/n_j-1)}\theta_j \ (\mathrm{EXPN})$$

$$= 0 \qquad (3.13)$$

where the expression EXPN is given by

$$\mathrm{EXPN} =$$

$$\begin{aligned}
[a_i\, a_j \{ & (t_{i_{1,4}} - t_{j_{1,4}})(t_{i_{2,1}} t_{j_{3,1}} - t_{i_{3,1}} t_{j_{2,1}}) \\
& + (t_{i_{2,4}} - t_{j_{2,4}})(t_{i_{3,1}} t_{j_{1,1}} - t_{i_{1,1}} t_{j_{3,1}}) \\
& + (t_{i_{3,4}} - t_{j_{3,4}})(t_{i_{1,1}} t_{j_{2,1}} - t_{i_{2,1}} t_{j_{1,1}})\} \sin^{(2-2/n_i)}\theta_i \\
+ b_i\, a_j \{ & (t_{i_{1,4}} - t_{j_{1,4}})(t_{i_{3,2}} t_{j_{2,1}} - t_{i_{2,2}} t_{j_{3,1}}) \\
& + (t_{i_{2,4}} - t_{j_{2,4}})(t_{i_{1,2}} t_{j_{3,2}} - t_{i_{3,2}} t_{j_{1,1}}) \\
& + (t_{i_{3,4}} - t_{j_{3,4}})(t_{i_{2,2}} t_{j_{1,2}} - t_{i_{1,2}} t_{j_{2,1}})\} \cos^{(2-2/n_i)}\theta_i
\end{aligned}$$

Eqn. (3.14) contd.....

$$+ a_i\ a_j\ b_i\ \{\ t_{i_{1,2}}\ (t_{i_{2,1}}\ t_{j_{3,1}} - t_{i_{3,1}}\ t_{j_{2,1}})$$

$$+ t_{i_{2,2}}\ (t_{i_{3,1}}\ t_{j_{1,1}} - t_{i_{1,1}}\ t_{j_{3,1}})$$

$$+ t_{i_{3,2}}\ (t_{i_{1,1}}\ t_{j_{2,1}} - t_{i_{2,1}}\ t_{j_{1,1}})\}\ ]\ \sin^{(2-2/n_j)}\theta_j$$

$$+[a_i\ b_j\{\ (t_{i_{1,4}} - t_{j_{1,4}})(t_{i_{3,1}}\ t_{j_{2,2}} - t_{i_{2,1}}\ t_{j_{3,2}})$$

$$+ (t_{i_{2,4}} - t_{j_{2,4}})(t_{i_{1,1}}\ t_{j_{3,2}} - t_{i_{3,1}}\ t_{j_{1,2}})$$

$$+ (t_{i_{3,4}} - t_{j_{3,4}})(t_{i_{2,1}}\ t_{j_{1,2}} - t_{i_{1,1}}\ t_{j_{2,2}})\}\ \sin^{(2-2/n_i)}\theta_i$$

$$+ b_i\ b_j\{\ (t_{i_{1,4}} - t_{j_{1,4}})(t_{i_{2,2}}\ t_{j_{3,2}} - t_{i_{3,2}}\ t_{j_{2,2}})$$

$$+ (t_{i_{2,4}} - t_{j_{2,4}})(t_{i_{3,2}}\ t_{j_{1,2}} - t_{i_{1,2}}\ t_{j_{3,2}})$$

$$+ (t_{i_{3,4}} - t_{j_{3,4}})(t_{i_{1,2}}\ t_{j_{2,2}} - t_{i_{2,2}}\ t_{j_{1,2}})\}\ \cos^{(2-2/n_i)}\theta_i$$

$$+ a_i\ b_i\ b_j\{\ t_{i_{1,2}}\ (t_{i_{3,1}}\ t_{j_{2,2}} - t_{i_{2,1}}\ t_{j_{3,2}})$$

$$+ t_{i_{2,2}}\ (t_{i_{1,1}}\ t_{j_{3,2}} - t_{i_{3,1}}\ t_{j_{1,2}})$$

$$+ t_{i_{3,2}}\ (t_{i_{2,1}}\ t_{j_{1,2}} - t_{i_{1,1}}\ t_{j_{2,2}})\}\ ]\ \cos^{(2-2/n_j)}\theta_j$$

$$- a_i\ a_j\ b_j\{\ t_{j_{1,1}}\ (t_{i_{3,1}}\ t_{j_{2,2}} - t_{i_{2,1}}\ t_{j_{3,2}})$$

$$+ t_{j_{2,1}}\ (t_{i_{1,1}}\ t_{j_{3,2}} - t_{i_{3,1}}\ t_{j_{1,2}})$$

$$+ t_{j_{3,1}}\ (t_{i_{2,1}}\ t_{j_{1,2}} - t_{i_{1,1}}\ t_{j_{2,2}})\}\ \sin^{(2-2/n_i)}\theta_i$$

Eqn. (3.14) contd......

$$- a_j \; b_i \; b_j \; \{ \; t_{j_{1,1}} \; (t_{i_{2,2}} \; t_{j_{3,2}} - t_{i_{3,2}} \; t_{j_{2,2}})$$

$$+ t_{j_{2,1}} \; (t_{i_{3,2}} \; t_{j_{1,2}} - t_{i_{1,2}} \; t_{j_{3,2}})$$

$$+ t_{j_{3,1}} \; (t_{i_{1,2}} \; t_{j_{2,2}} - t_{i_{2,2}} \; t_{j_{1,2}}) \} \quad \cos^{(2-2/n_i)}\theta_i$$

$$\ldots\ldots \qquad (3.14)$$

But

$$(\frac{2}{n_i})(\frac{2}{n_j})\cos^{(2/n_i-1)}\theta_i \; \sin^{(2/n_i-1)}\theta_i \; \cos^{(2/n_j-1)}\theta_j \; \sin^{(2/n_j-1)}\theta_j \neq 0$$

for all values of θ_i and θ_j and $n_i \geq 2$ and $n_j \geq 2$. So if the values of n_i and n_j are kept equal to or greater than 2, the condition for developability reduces to

$$\text{EXPN} = 0 \qquad (3.15)$$

where the expression EXPN is given by Eqn. (3.14).

Further, if the spatial configuration of the super-convolute is given as per definition A, then substituting for the elements of the matrices $[T_i]$ and $[T_j]$, Eqn. (3.14) reduces to

$$\text{EXPN} = [a_i \; a_j \; \{\sin\,(\alpha_j - \alpha_i)(d_j \; \cos\gamma_i \; \cos\beta_j \; \sin\gamma_j$$

$$- di \; \cos\beta_i \; \sin\gamma_i \; \cos\gamma_j)$$

$$+ \cos\,(\alpha_j - \alpha_i)(d_j \; \sin\beta_i \; \sin\gamma_i \; \cos\beta_j \; \sin\gamma_j$$

$$+ d_i \; \cos\beta_i \; \sin\gamma_i \; \sin\beta_j \; \sin\gamma_j)$$

$$- d_j \; \cos\beta_i \; \sin\gamma_i \; \sin\beta_j \; \sin\gamma_j$$

$$- d_i \; \sin\beta_i \; \sin\gamma_i \; \cos\beta_j \; \sin\gamma_j \} \quad \sin^{(2-2/n_i)}\theta_i$$

Eqn. (3.16) contd.....

$$+ b_i\, a_j \{\sin(\alpha_j - \alpha_i)(d_j \sin\gamma_i \cos\beta_j \sin\gamma_j$$

$$+ d_i \cos\beta_i \cos\gamma_i \cos\gamma_j)$$

$$- \cos(\alpha_j - \alpha_i)(d_j \sin\beta_i \cos\gamma_i \cos\beta_j \sin\gamma_j$$

$$+ d_i \cos\beta_i \cos\gamma_i \sin\beta_j \sin\gamma_j)$$

$$+ d_j \cos\beta_i \cos\gamma_i \sin\beta_j \sin\gamma_j$$

$$+ d_i \sin\beta_i \cos\gamma_i \cos\beta_j \sin\gamma_j \} \cos(2-2/n_i)\theta_i$$

$$+ a_i\, a_j\, b_i \{\sin(\alpha_j - \alpha_i) \cos\beta_i \cos\gamma_j$$

$$- \cos(\alpha_j - \alpha_i) \cos\beta_i \sin\beta_j \sin\gamma_j$$

$$+ \sin\beta_i \cos\beta_j \sin\gamma_j \}] \sin(2-2/n_j)\theta_j$$

$$+[a_i\, b_j \{ -\sin(\alpha_j - \alpha_i)(d_j \cos\gamma_i \cos\beta_j \cos\gamma_j$$

$$+ d_i \cos\beta_i \sin\gamma_i \sin\gamma_j)$$

$$- \cos(\alpha_j - \alpha_i)(d_j \sin\beta_i \sin\gamma_i \cos\beta_j \cos\gamma_j$$

$$+ d_i \cos\beta_i \sin\gamma_i \sin\beta_j \cos\gamma_j)$$

$$+ d_j \cos\beta_i \sin\gamma_i \sin\beta_j \cos\gamma_j$$

$$+ d_i \sin\beta_i \sin\gamma_i \cos\beta_j \cos\gamma_j \} \sin(2-2/n_i)\theta_i$$

$$+ b_i\, b_j \{ \sin(\alpha_j - \alpha_i)(-d_j \sin\gamma_i \cos\beta_j \cos\gamma_j$$

$$+ d_i \cos\beta_i \cos\gamma_i \sin\gamma_j)$$

$$+ \cos(\alpha_j - \alpha_i)(d_j \sin\beta_i \cos\gamma_i \cos\beta_j \cos\gamma_j$$

$$+ d_i \cos\beta_i \cos\gamma_i \sin\beta_j \cos\gamma_j)$$

$$- d_j \cos\beta_i \cos\gamma_i \sin\beta_j \cos\gamma_j$$

$$- d_i \sin\beta_i \cos\gamma_i \cos\beta_j \cos\gamma_j \} \cos(2-2/n_i)\theta_i$$

Eqn. (3.16) Contd.....

$$+ a_i\, b_i\, b_j \{ \sin(\alpha_j - \alpha_i) \cos\beta_i \sin\gamma_j$$

$$+ \cos(\alpha_j - \alpha_i) \cos\beta_i \sin\beta_j \cos\gamma_j$$

$$- \sin\beta_i \cos\beta_j \cos\gamma_j \; \}] \cos^{(2-2/n_j)}\theta_j$$

$$- a_i\, a_j\, b_j \{ \sin(\alpha_j - \alpha_i) \cos\gamma_i \cos\beta_j$$

$$+ \cos(\alpha_j - \alpha_i) \sin\beta_i \sin\gamma_i \cos\beta_j$$

$$- \cos\beta_i \sin\gamma_i \sin\beta_j \; \} \sin^{(2-2/n_i)}\theta_i$$

$$- a_j\, b_i\, b_j \{ \sin(\alpha_j - \alpha_i) \sin\gamma_i \cos\beta_j$$

$$- \cos(\alpha_j - \alpha_i) \sin\beta_i \cos\gamma_i \cos\beta_j$$

$$+ \cos\beta_i \cos\gamma_i \sin\beta_j \; \} \cos^{(2-2/n_i)}\theta_i \qquad .. \qquad (3.16)$$

3.4.3 The $k\underline{b}$ Vector

The $k\underline{b}$ vector at the generic point of the primary directrix is given by

$$k\underline{b} = \frac{\dot{\underline{r}}_i \times \ddot{\underline{r}}_i}{\dot{s}_i^{\,3}} \qquad (3.17)$$

Substituting for $\dot{\underline{r}}_i$, $\ddot{\underline{r}}_i$ and $\dot{s}_i$ and simplifying the expression, the $k\underline{b}$ vector is obtained as

$$k\underline{b} = \frac{(n_i-1)\; a_i\; b_i\; \sin^{(2-4/n_i)}\theta_i \;\cos^{(2-4/n_i)}\theta_i}{(\text{Factor } 1)_i^3} \begin{bmatrix} t_{i_{2,1}} t_{i_{3,2}} - t_{i_{2,2}} t_{i_{3,1}} \\ t_{i_{3,1}} t_{i_{1,2}} - t_{i_{3,2}} t_{i_{1,1}} \\ t_{i_{1,1}} t_{i_{2,2}} - t_{i_{1,2}} t_{i_{2,1}} \end{bmatrix} \quad \ldots \quad (3.18)$$

where $(\text{Factor } 1)_i$ is given by Eqn. (3.11).

If the spatial configuration of the super convolute is given as per definition A, then Eqn. (3.18) reduces to

$$k\underline{b} = \frac{(n_i-1) a_i\; b_i\; \sin^{(2-4/n_i)}\theta_i \;\cos^{(2-4/n_i)}\theta_i}{(\text{Factor } 1)_i^3} \begin{bmatrix} \sin\alpha_i \cos\beta_i \\ -\cos\alpha_i \cos\beta_i \\ -\sin\beta_i \end{bmatrix} \quad \ldots \quad (3.19)$$

3.4.4 Magnitude of the Geodesic Curvature

The magnitude of the geodesic curvature is given by

$$k_g = \underline{n}_s \cdot k\underline{b} \qquad (3.20)$$

where $\underline{n}_s$ is the unit normal vector to the surface of the super-convolute.

$$\underline{n}_s = \frac{\dot{\underline{r}}_i \times (\underline{r}_j - \underline{r}_i)}{\|\dot{\underline{r}}_i \times (\underline{r}_j - \underline{r}_i)\|} \qquad (3.21)$$

3.4.5 Arc Length

The arc length between two consecutive positions of the generic point along the primary directrix is given by

$$s_i = \int_{\theta_1}^{\theta_2} \frac{2}{n_i} [\{ a_i \cos^{(2/n_i - 1)}\theta_i \sin\theta_i \}^2 + \{ b_i \sin^{(2/n_i - 1)}\theta_i \cos\theta_i \}^2]^{1/2} d\theta_i \qquad (3.22)$$

where θ_1 and θ_2 are the values of the parameter θ_i corresponding to the two consecutive positions.

3.4.6 Arc-tangent Angle

The angle Ψ_i made by the tangent vector at a generic point along the primary directrix with respect to the tangent vector at the starting position of the generic point is given by

$$\Psi_i = \int_0^{s_i} k\,(s_i)\, ds_i \qquad (3.23)$$

where $k\,(s_i)$ indicate that the curvature vector is a function of the arc length s_i.

3.4.7 Angle Between the Arc-tangent and the Generatrix

The angle between the arc-tangent and the generatrix is given by

$$\phi_i = \cos^{-1} \left[\frac{\dot{\underline{r}}_i \cdot (\underline{r}_j - \underline{r}_i)}{\| \dot{\underline{r}}_i \| \, \| (\underline{r}_j - \underline{r}_i) \|} \right] . \qquad (3.24)$$

3.4.8 Algorithm for Development of Super-Conical Convolute

The algorithm for the development of the super-conical convolute is given below.

Step 1 : (a) Read the data about the spatial configuration of the super-conical convolute. If the spatial configuration is given as per definition (A) (refer to Section 3.3.1), then read the values of α_i , β_i , γ_i , d_i , α_j , β_j , γ_j and d_j. If the spatial configuration is given as per definition (B) (refer to Section 3.3.2), then read the values of γ_i , γ_j and the elements of the matrices $[MTH_i]$ and $[MTH_j]$ where

$$[MTH_i] = \begin{bmatrix} n_{c_{i_x}} & b_{i_x} & t_{i_x} & r_{o_{i_x}} \\ n_{c_{i_y}} & b_{i_y} & t_{i_y} & r_{o_{i_y}} \\ n_{c_{i_z}} & b_{i_z} & t_{i_z} & r_{o_{i_z}} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the matrix $[MTH_j]$ is given by replacing the subscript i by j in the above equation.

(b) Read the values of a_i , b_i , n_i , a_j , b_j and n_j.

(c) Also read the value of NQ , the number of parts into which $\pi/2$ radians are to be divided. The total number of points to be considered along the directrices is then

$$N = 4NQ + 1 \quad .$$

Step 2 : Find the transformation matrix for each directrix. Use Eqn. (3.5) if the spatial configuration is given as per definition (A) and use Eqn. (3.6) if it is given as per definition (B).

Step 3 : Find $\theta_{i_{INC}}$, the increment to be used for the parameter θ_i.

$$\theta_{i_{INC}} = 2\pi/(N-1) \text{ radians.}$$

Step 4 : Calculate the various terms in Eqn. (3.14) which are constant for the given convolute, i.e. independent of the values of θ_i and θ_j.

Step 5 : Set the initial conditions for the integration of Eqn. (3.22) for arc length, Eqn. (3.23) for arc-tangent angle and Serret-Frenet equations for the development of the primary directrix.

$$s_i\ (1) = 0$$
$$\Psi_i\ (1) = 0$$
$$Y_1\ (1) = 0$$
$$Y_2\ (1) = 1.0$$
$$Y_3\ (1) = 0$$
$$Y_4\ (1) = 0$$

Also set $\theta_i\ (1) = 0$

Step 6 : Do the following steps (Step 7 to Step 18) for K = 1 to N.

Step 7 : If K = 1, go to Step 8. If K > 1, set

$$\theta_i\ (K) = \theta_i\ (K - 1) + \theta_{i_{INC}}\ .$$

Step 8 : For the given value of θ_i (K) find the corresponding value of θ_j (K) from the condition for developability. This fixes the positions of the generic points of the two directrices. Further calculations given in the following steps are carried out for this pair of positions only.

Step 9 : Calculate the unit tangent vectors at these points (use Eqn. (3.10)).

Step 10: Calculate the length of the generatrix.

$$L = \left\| (\underline{r}_j - \underline{r}_i) \right\|$$

Step 11: Calculate the angle between the arc-tangent and the generatrix, ϕ_i (K) from Eqn. (3.24).

Step 12: Calculate the unit normal vector to the convolute surface (Eqn. (3.21)).

Step 13: Calculate the curvature vector using Eqn. (3.18).

Step 14: Calculate the magnitude of geodesic curvature from Eqn. (3.20).

Step 15: If K = 1, go to Step 18. If K > 1, go to Step 16.

Step 16: Calculate the arc length from Eqn. (3.22). Suitable numerical method can be used for the integration.

$$s_i\ (K) = s_i\ (K-1)$$

$$+ \frac{2}{n_i} \int_{\theta_i(K-1)}^{\theta_i(K)} \left[\{a_i \cos^{(2/n_i-1)}\theta_i \sin\theta_i\}^2 + \{b_i \sin^{(2/n_i-1)}\theta_i \cos\theta_i\}^2 \right]^{\frac{1}{2}} d\theta_i$$

For values of θ_i (K) upto 90 degrees, the value of s_i (K) can be obtained by integration. For values of θ_i (K) above 90 degrees, the symmetry of the super-ellipse (the primary directrix) about its major and minor diameters can be utilized and the calculations simplified as shown below:

$$s_i\ (K) = 2s_i\ (N1) - s_i\ (N2-K+1) \qquad 90° < \theta_i(K) \leq 180°$$

$$s_i\ (K) = 2s_i\ (N1) + s_i\ (K-N2+1) \qquad 180° < \theta_i(K) \leq 270°$$

and $$s_i\ (K) = 4s_i\ (N1) - s_i\ (N-K+1) \qquad 270° < \theta_i(K) \leq 360°$$

where

$$N_1 = NQ + 1$$

$$N_2 = N1 + NQ$$

$$N_3 = N2 + NQ$$

and $$N = N3 + NQ.$$

Step 17: Carry out the integration of Serret-Frenet equations and Eqn. (3.23) to find Y_1 (K), Y_2 (K), Y_3 (K), Y_4 (K) and Ψ_i (K).

Step 18: The co-ordinates of the generic point in the development of the primary directrix are given by

$$x_{d_i}(K) = Y_1(K)$$

and $$y_{d_i}(K) = Y_3(K).$$

Corresponding co-ordinates of the generic point on the development of the secondary directrix are

$$x_{d_j}(K) = x_{d_i}(K) + L\cos\delta$$

and $$y_{d_j}(K) = y_{d_i}(K) + L\sin\delta$$

where

$$\delta = \phi_i(K) + \Psi_i(K).$$

Go to Step 6.

3.5 Case Studies

The foregoing methods for development of conical and super-conical convolutes can be illustrated by means of some examples. Based on the algorithm discussed in Section 3.4.8, a set of two computer programmes have been developed. For the first programme, the input data is as per definition (A) of the primary and secondary directrices. The second programme accepts the input data as per definition (B) of the directrices.

For each curve of the directrices the values of a, b and n can be selected. By making suitable combinations the following pair of directrices can be obtained. Circle-circle, circle-ellipse, ellipse-ellipse, super-ellipse - super-ellipse, etc. The orientation of the

planes of the two directrices are to be given. Also certain auxiliary data which govern the numerical accuracy of the integration process needs to be given.

Here conical or super-conical convolutes whose directrices are specified as per definition (A) are taken for example. Convolutes whose directrices are specified as per definition (B) are not presented here. Any of the convolutes that form part of the thin ducts discussed in Chapter 5 can be taken as an example for such a convolute.

In all these examples the input data regarding the geometry of the convolute is as per the details given in Table 3.1. The output data is presented in graphical form.

3.5.1 Example 3.1

The directrices of the convolute considered are circles. The planes of these directrices are perpendicular to the O-XY plane. The output data is presented in graphical form in Figure 3.6. Figure 3.6a gives the orthographic views of the convolute and Figure 3.6b the development of the convolute.

3.5.2 Example 3.2

One of the directrices is a cirrle and the other an ellipse. The planes of the directrices are inclined to the O-XY plane. The output data is presented in graphical form in Figure 3.7.

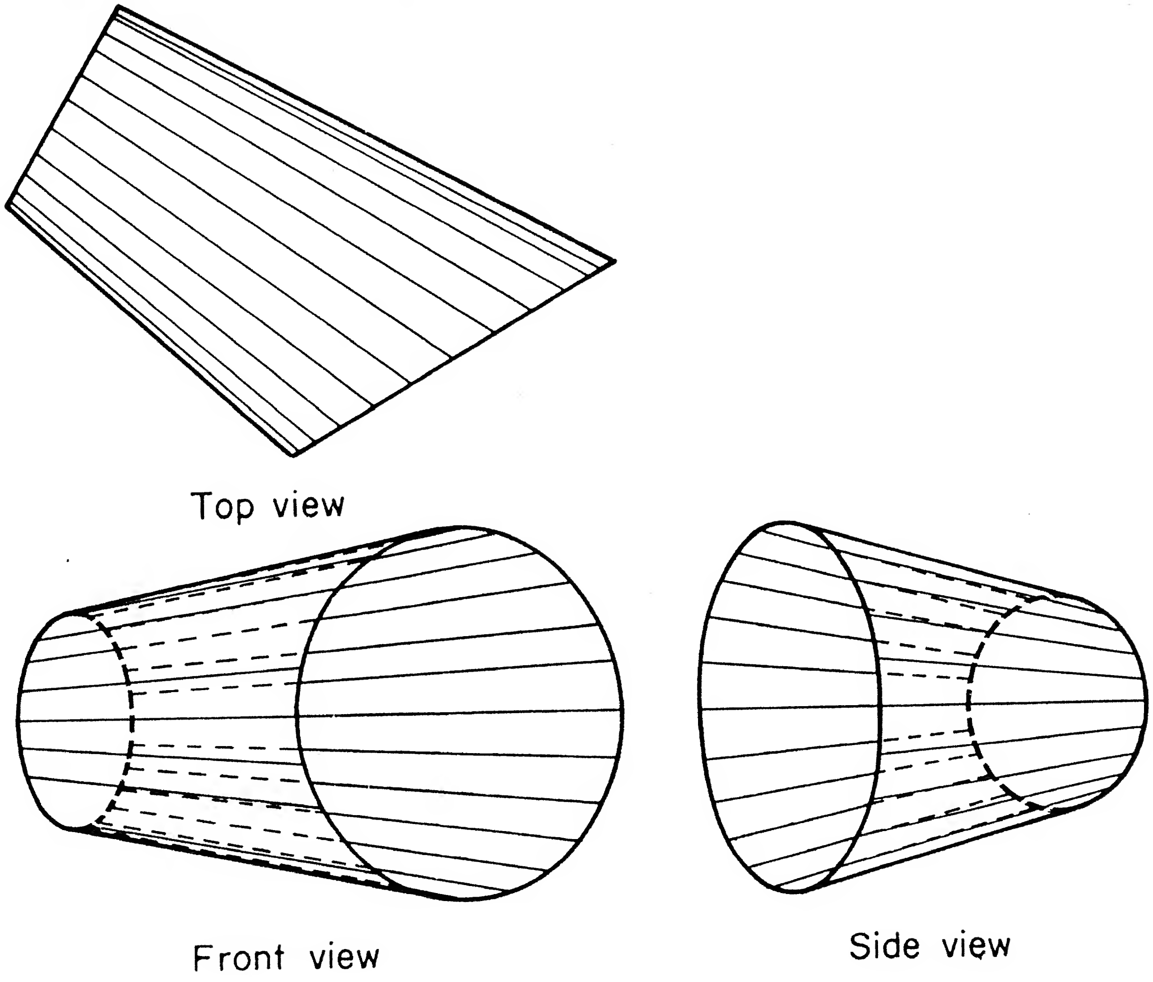


Fig. 3.6 a Orthographic views of conical convolute.
Example 3.1 .

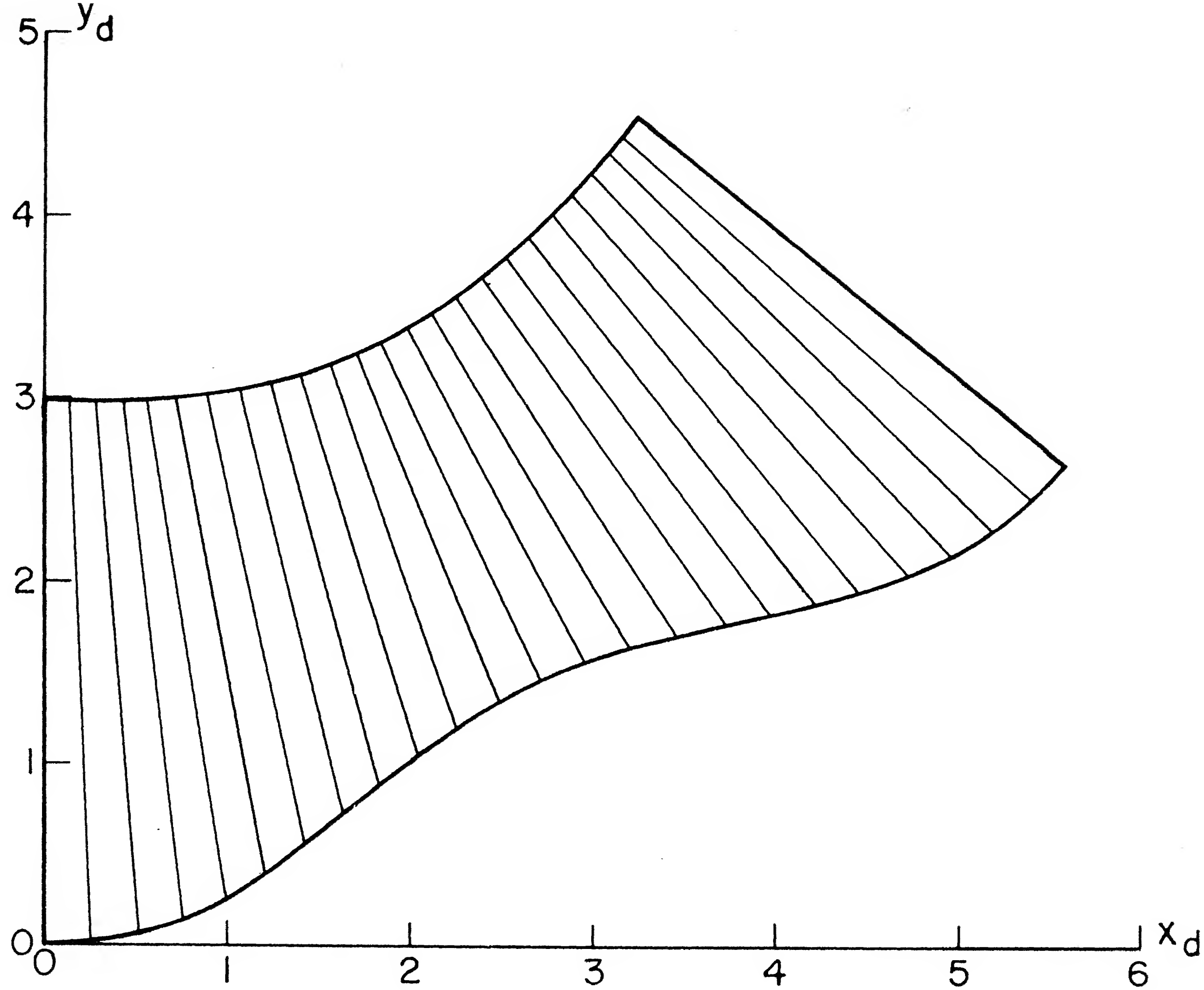


Fig. 3.6 b Development of conical convolute.
Example 3.1.

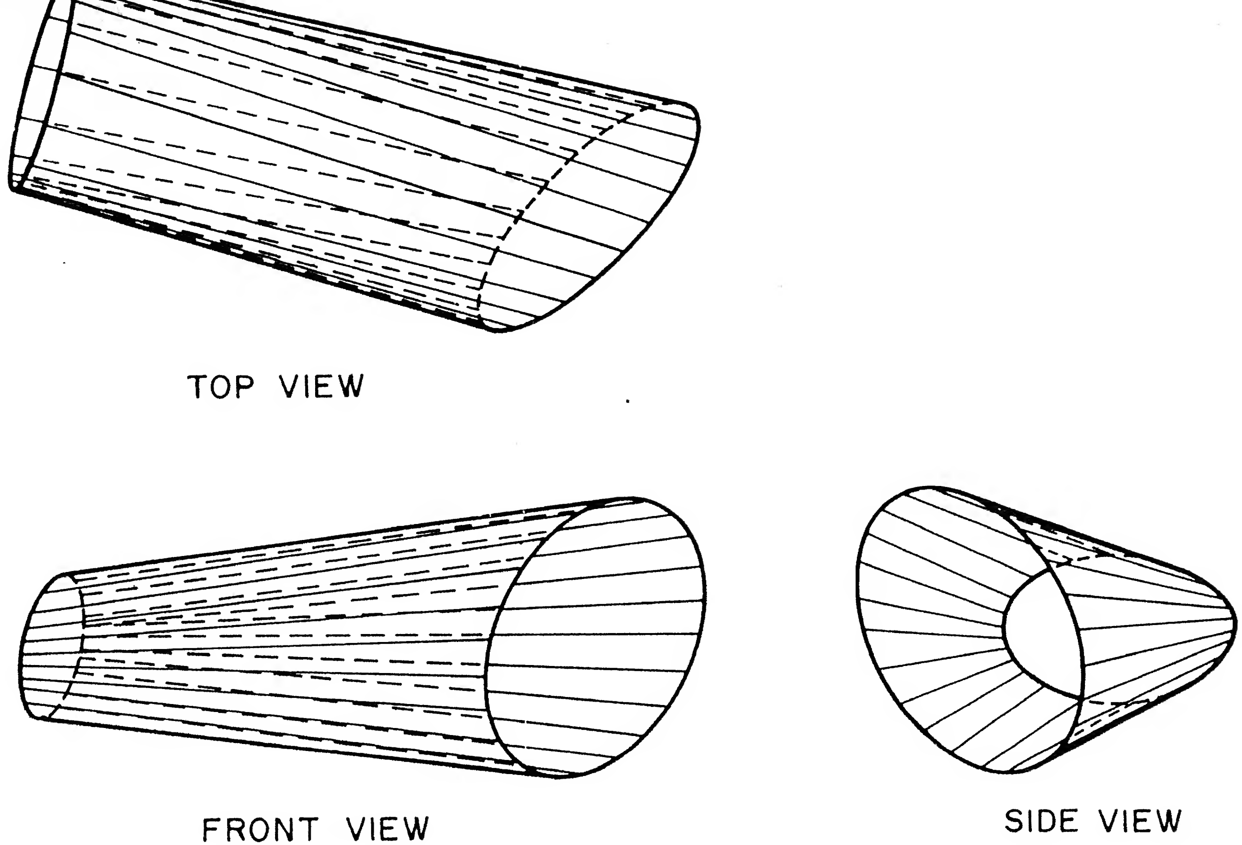


Fig.3.7a Orthographic views of conical convolute.
Example 3.2 .

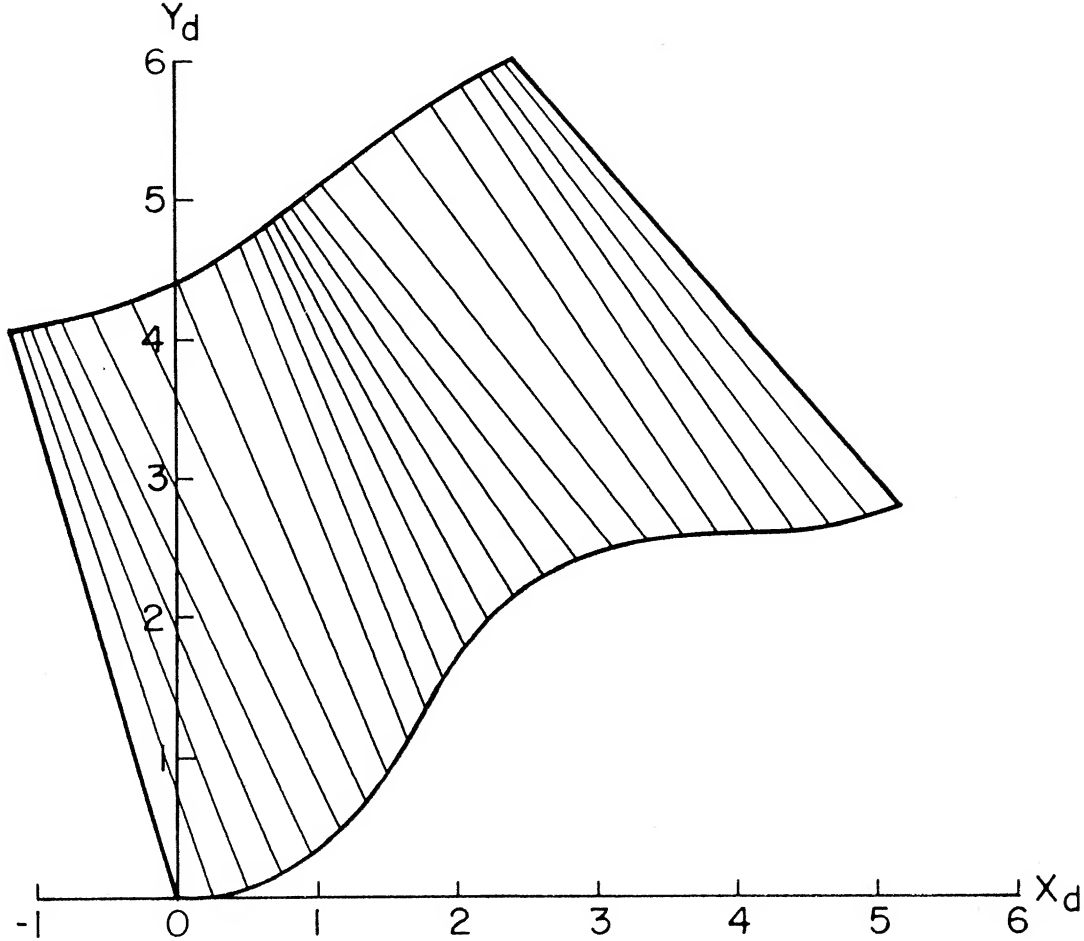


Fig. 3.7b Development of conical convolute. Example 3.2

3.5.3 Example 3.3

Both the directrices of the convolute are ellipses whose planes are inclined to the O-XY plane. The curves of the directrices have also been rotated about the respective local z axes. The output data is given in graphical form in Figure 3.8.

3.5.4 Example 3.4

Both the directrices of the convolute are super-ellipses. Their planes are inclined to the O-XY plane. Also the curves of the directrices are rotated about the respective local z axes. The output data is given in Table 3.2 and 3.3 and is also graphically presented in Figure 3.9. Following details are given in Table 3.2:

(i) the value of the parameter θ_i of the primary directrix,

(ii) the corresponding value of the parameter θ_j of the secondary directrix,

(iii) the co-ordinates of the generic point on the primary directrix, (x_i , y_i , z_i),

(iv) the co-ordinates of the corresponding generic point on the secondary directrix, (x_j , y_j , z_j),

(v) the magnitude of geodesic curvature, kg, of the primary directrix at (x_i , y_i , z_i),

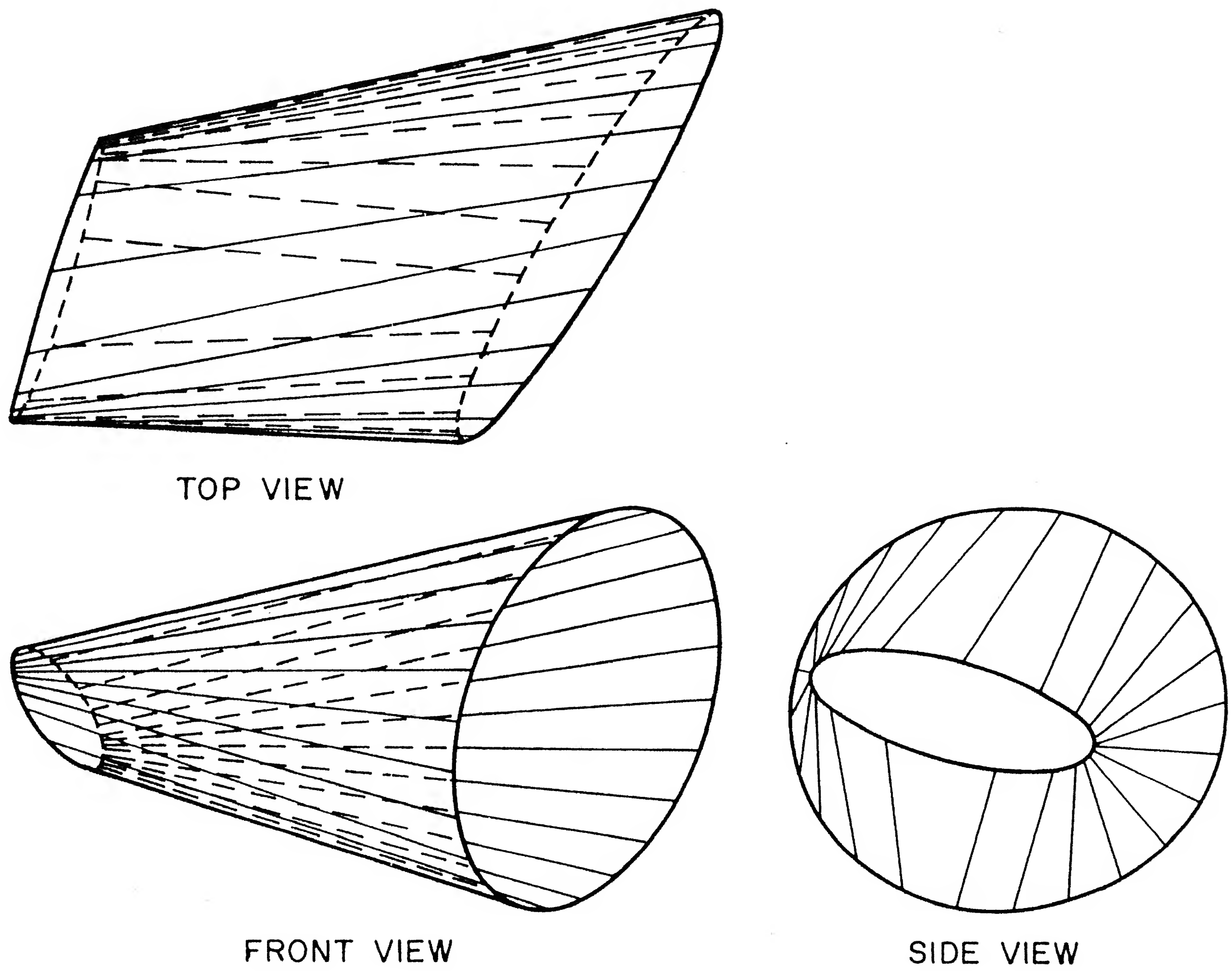


Fig. 3.8 a Orthographic views of conical convolute.
Example 3.3 .

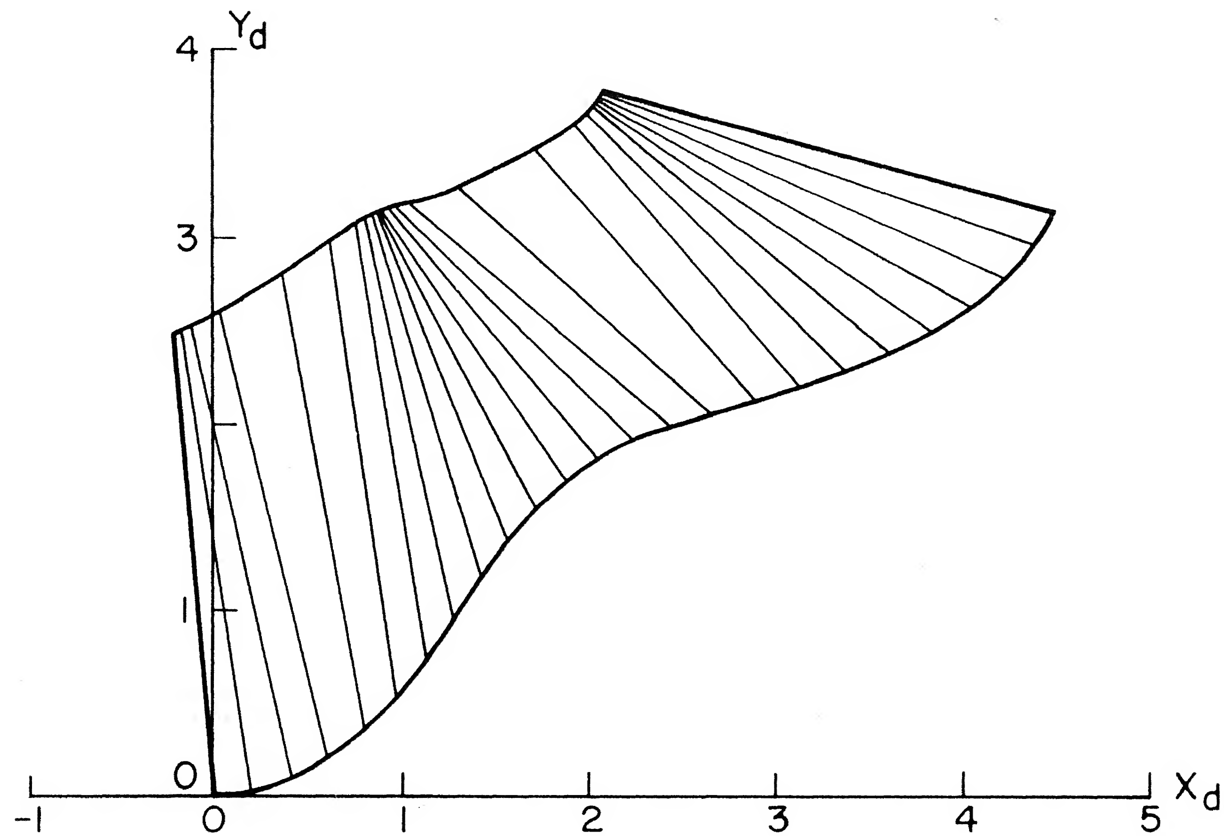


Fig. 3.8 b Development of conical convolute.
Example 3.3.

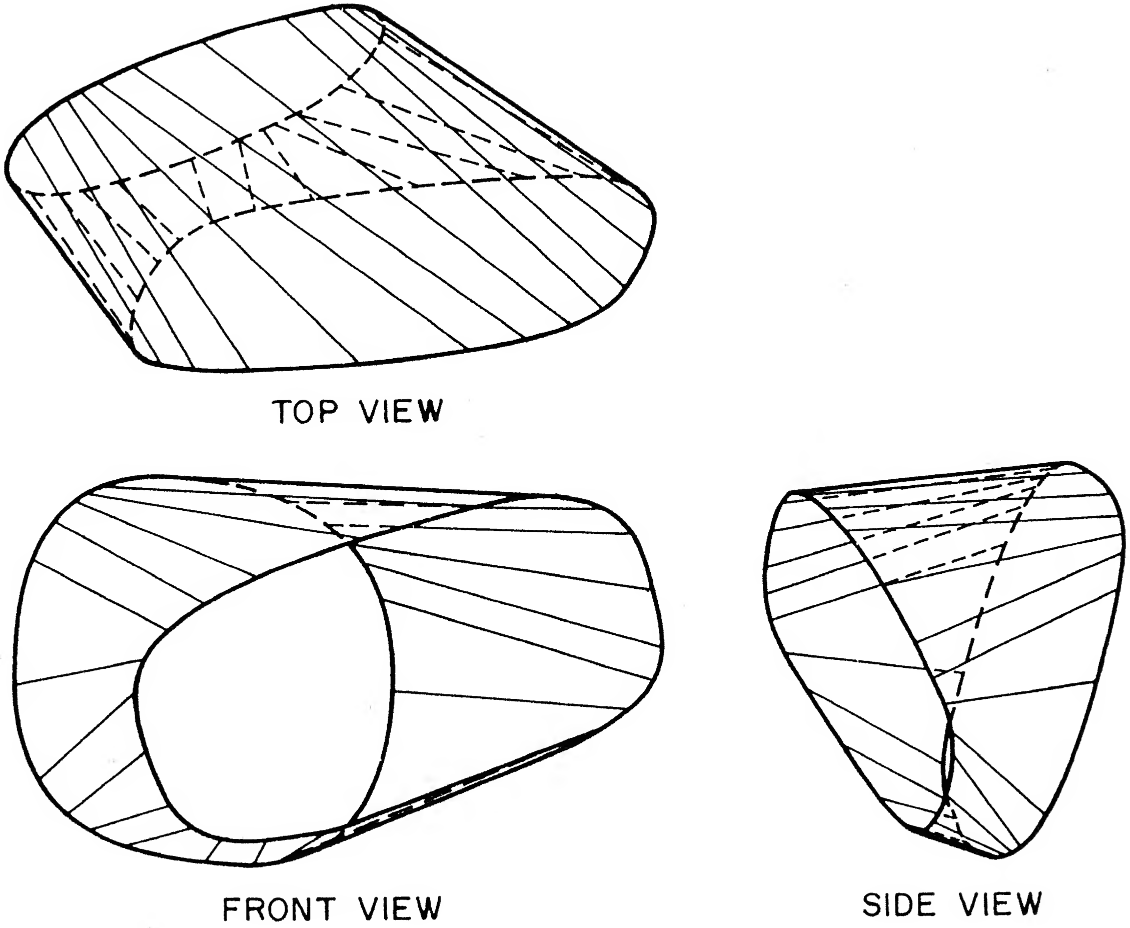


Fig.3.9a Orthographic views of super-conical convolute.
Example 3.4.

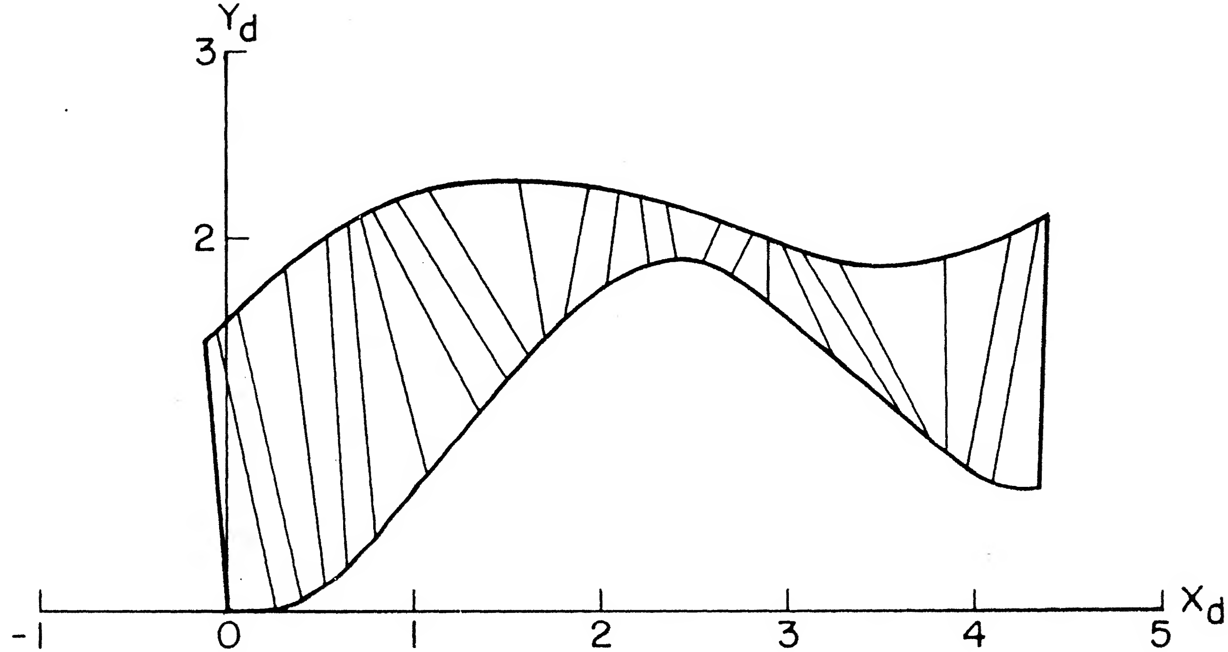


Fig. 3.9b Development of super-conical convolute.
Example 3.4.

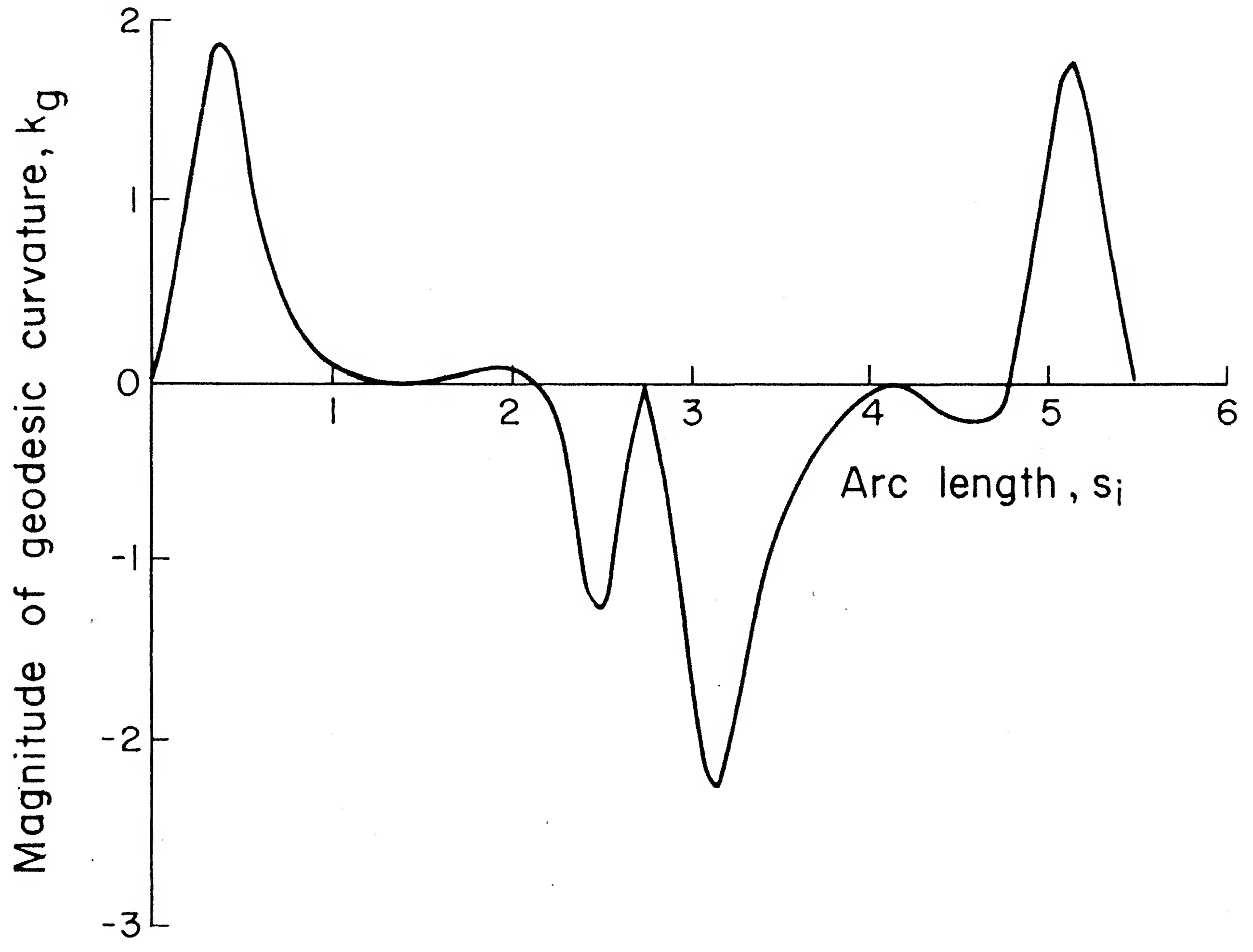


Fig. 3.9c Magnitude of geodesic curvature vs. arc length. Example 3.4.

(vi) the arc length of primary directrix, s_i , from the starting point to the point (x_i , y_i , z_i) and

(vii) the length of the generatrix, L.

In Table 3.3 the co-ordinates of the development of the primary directrix (x_{d_i} , y_{d_i}) and of the secondary directrix (x_{d_j} , y_{d_j}) are given.

Table 3.1 Input Data about Geometry of Directrices

Example No.		3.1	3.2	3.3	3.4
Primary directrix		Circle	Circle	Ellipse	Super-ellipse
Secondary directrix		Circle	Ellipse	Ellipse	Super-ellipse
Details of Primary Directrix	α_i , deg	30.	45.	60.	15.
	β_i , deg	0.	20.	10.	30.
	γ_i , deg	0.	0.	15.	20.
	d_i	5.	7.5	7.5	10.
	a_i	1.	1.	1.	1.
	b_i	1.	1.	0.8	0.6
	n_i	2.	2.	2.	3.25
Details of Secondary Directrix	α_j , deg	60.	75.	75.	20.
	β_j , deg	0.	10.	-10.	-15.
	γ_j , deg	0.	0.	-15.	-10.
	d_j	4.5	6.5	6.5	9.5
	a_j	0.6	0.8	0.6	0.75
	b_j	0.6	0.5	0.2	0.75
	n_j	2.	2.	2.	2.5

Table 3.2 Details of Super-Conical Convolute. Example 3.4

S. No.	θ_i deg	θ_j deg	x_i	y_i	z_i	x_j	y_j	z_j	k_g	s_i	L
1.	0.	43.839	10.611	2.666	0.296	9.477	3.586	0.478	0.	0.	1.471
2.	5.	50.630	10.581	2.594	0.404	9.464	3.585	0.493	0.630	0.134	1.497
3.	10.	53.625	10.560	2.555	0.460	9.440	3.583	0.517	1.091	0.205	1.522
4.	15.	57.250	10.537	2.523	0.502	9.408	3.580	0.544	1.483	0.263	1.546
5.	20.	61.448	10.511	2.495	0.537	9.369	3.574	0.573	1.769	0.314	1.571
6.	25.	66.047	10.482	2.470	0.566	9.323	3.565	0.603	1.904	0.362	1.595
7.	30.	71.104	10.449	2.447	0.590	9.268	3.553	0.631	1.871	0.409	1.619
8.	35.	76.620	10.412	2.426	0.609	9.202	3.537	0.659	1.697	0.455	1.644
9.	40.	82.708	10.372	2.406	0.623	9.119	3.514	0.685	1.436	0.503	1.673
10.	45.	90.802	10.327	2.388	0.634	8.961	3.466	0.718	1.147	0.552	1.742
11.	50.	97.980	10.278	2.371	0.640	8.838	3.426	0.734	0.869	0.604	1.787
12.	55.	103.610	10.225	2.356	0.642	8.761	3.399	0.737	0.632	0.660	1.800
13.	60.	108.553	10.166	2.341	0.640	8.699	3.376	0.734	0.444	0.720	1.797
14.	65.	112.922	10.102	2.328	0.634	8.649	3.355	0.727	0.303	0.786	1.782
15.	70.	116.834	10.031	2.315	0.623	8.606	3.337	0.718	0.200	0.859	1.756

Contd....

Table 3.2 Details of Super-Conical Convolute. Example 3.4

S. No.	θ_i deg	θ_j deg	x_i	y_i	z_i	x_j	y_j	z_j	k_g	s_i	L
16.	75.	120.287	9.951	2.304	0.607	8.570	3.321	0.707	0.126	0.941	1.718
17.	80.	123.281	9.859	2.292	0.535	8.540	3.307	0.696	0.074	1.037	1.668
18.	85.	125.646	9.746	2.281	0.553	8.517	3.296	0.686	0.036	1.154	1.599
19.	90.	127.036	9.534	2.263	0.488	8.504	3.289	0.680	0.000	1.377	1.466
20.	95.	128.369	9.322	2.246	0.421	8.492	3.283	0.673	0.030	1.600	1.352
21.	100.	130.504	9.211	2.239	0.383	8.473	3.273	0.662	0.047	1.718	1.300
22.	105.	133.098	9.122	2.236	0.349	8.451	3.261	0.648	0.058	1.813	1.261
23.	110.	136.150	9.047	2.235	0.317	8.426	3.246	0.629	0.063	1.895	1.227
24.	115.	139.602	8.981	2.236	0.285	8.400	3.230	0.605	0.062	1.968	1.195
25.	120.	143.514	8.923	2.239	0.254	8.372	3.212	0.576	0.051	2.034	1.163
26.	125.	147.998	8.872	2.245	0.221	8.343	3.190	0.540	0.026	2.094	1.129
27.	130.	153.170	8.828	2.253	0.189	8.313	3.166	0.492	-0.021	2.150	1.092
28.	135.	159.144	8.789	2.262	0.155	8.284	3.138	0.431	-0.099	2.202	1.048
29.	140.	166.264	8.756	2.274	0.121	8.256	3.104	0.349	-0.217	2.252	0.996
30.	145.	175.447	8.728	2.288	0.085	8.231	3.058	0.219	-0.385	2.299	0.927

Contd......

Table 3.2 Details of Super-Conical Convolute. Example 3.4

S. No.	θ_i deg	θ_j deg	x_i	y_i	z_i	x_j	y_j	z_j	k_g	s_i	L
31.	150.	189.270	8.706	2.304	0.048	8.216	2.979	-0.041	-0.642	2.345	0.838
32.	155.	200.515	8.690	2.323	0.009	8.233	2.943	-0.189	-0.967	2.392	0.795
33.	160.	210.500	8.678	2.345	-0.033	8.268	2.923	-0.303	-1.211	2.440	0.759
34.	165.	218.995	8.673	2.370	-0.077	8.311	2.914	-0.390	-1.257	2.491	0.724
35.	170.	225.828	8.674	2.400	-0.127	8.355	2.912	-0.453	-1.069	2.549	0.685
36.	175.	230.828	8.682	2.438	-0.187	8.392	2.913	-0.495	-0.676	2.620	0.636
37.	180.	233.536	8.707	2.510	-0.296	8.414	2.915	-0.516	-0.000	2.754	0.546
38.	185.	235.672	8.737	2.583	-0.404	8.432	2.917	-0.533	-0.716	2.888	0.471
39.	190.	238.838	8.759	2.622	-0.460	8.460	2.921	-0.556	-1.254	2.959	0.433
40.	195.	242.406	8.782	2.653	-0.502	8.494	2.926	-0.580	-1.720	3.017	0.404
41.	200.	246.317	8.808	2.681	-0.537	8.534	2.934	-0.604	-2.069	3.069	0.379
42.	205.	250.515	8.837	2.706	-0.566	8.580	2.944	-0.628	-2.252	3.117	0.356
43.	210.	254.942	8.870	2.729	-0.590	8.632	2.956	-0.651	-2.249	3.163	0.334
44.	215.	259.770	8.906	2.750	-0.609	8.694	2.972	-0.673	-2.091	3.209	0.314
45.	220.	265.114	8.947	2.770	-0.623	8.772	2.995	-0.694	-1.340	3.257	0.294

Contd......

Table 3.2 Details of Super-Conical Convolute. Example 3.4

S. No.	θ_i deg	θ_j deg	x_i	y_i	z_i	x_j	y_j	z_j	k_g	s_i	L
46.	225.	272.119	8.991	2.788	-0.634	8.921	3.041	-0.722	-1.569	3.306	0.277
47.	230.	277.234	9.040	2.805	-0.640	9.005	3.069	-0.733	-1.321	3.358	0.282
48.	235.	281.031	9.094	2.821	-0.642	9.059	3.087	-0.736	-1.088	3.414	0.285
49.	240.	284.083	9.153	2.835	-0.640	9.099	3.102	-0.737	-0.877	3.475	0.289
50.	245.	286.676	9.217	2.849	-0.634	9.132	3.114	-0.736	-0.693	3.541	0.297
51.	250.	288.869	9.288	2.861	-0.623	9.159	3.124	-0.734	-0.534	3.614	0.313
52.	255.	290.890	9.368	2.873	-0.607	9.182	3.134	-0.731	-0.395	3.696	0.343
53.	260.	292.681	9.459	2.884	-0.585	9.203	3.142	-0.728	-0.270	3.791	0.391
54.	265.	294.186	9.572	2.895	-0.553	9.220	3.149	-0.724	-0.150	3.909	0.467
55.	270.	295.238	9.785	2.914	-0.488	9.231	3.154	-0.722	0.000	4.131	0.647
56.	275.	296.566	9.996	2.930	-0.421	9.246	3.160	-0.719	-0.141	4.354	0.840
57.	280.	299.045	10.107	2.937	-0.383	9.272	3.172	-0.711	-0.227	4.472	0.928
58.	285.	302.670	10.196	2.941	-0.349	9.308	3.189	-0.698	-0.278	4.567	0.986
59.	290.	307.727	10.272	2.942	-0.317	9.356	3.213	-0.676	-0.273	4.649	1.020
60.	295.	314.503	10.337	2.940	-0.285	9.415	3.244	-0.639	-0.184	4.722	1.034

Contd....

Table 3.2 Details of Super-Conical Convolute. Example 3.4

S. No.	θ_i deg	θ_j deg	x_i	y_i	z_i	x_j	y_j	z_j	k_g	s_i	L
61.	300.	323.342	10.395	2.937	-0.254	9.481	3.286	-0.578	0.004	4.788	1.031
62.	305.	334.186	10.446	2.931	-0.221	9.546	3.337	-0.483	0.276	4.848	1.021
63.	310.	346.692	10.491	2.924	-0.189	9.600	3.396	-0.343	0.598	4.904	1.020
64.	315.	362.005	10.530	2.914	-0.155	9.636	3.485	-0.077	0.927	4.957	1.063
65.	320.	372.906	10.563	2.902	-0.121	9.636	3.533	0.092	1.232	5.006	1.141
66.	325.	380.771	10.590	2.888	-0.085	9.621	3.556	0.192	1.512	5.053	1.210
67.	330.	387.203	10.612	2.872	-0.048	9.600	3.570	0.267	1.714	5.099	1.269
68.	335.	392.604	10.629	2.853	-0.009	9.577	3.579	0.325	1.782	5.146	1.321
69.	340.	397.261	10.640	2.831	0.033	9.553	3.583	0.373	1.685	5.194	1.365
70.	345.	401.172	10.646	2.806	0.077	9.530	3.586	0.410	1.434	5.246	1.401
71.	350.	404.510	10.645	2.776	0.127	9.508	3.587	0.441	1.070	5.304	1.431
72.	355.	407.161	10.637	2.739	0.187	9.490	3.586	0.464	0.626	5.375	1.453
73.	360.	408.838	10.611	2.666	0.296	9.477	3.586	0.478	0.000	5.508	1.471

Table 3.3 Development of Super-Conical Convolute. Example 3.4

S. No.	x_{d_i}	y_{d_i}	x_{d_j}	y_{d_j}
1.	0.	0.	-0.114	1.467
2.	0.134	0.002	-0.106	1.479
3.	0.205	0.007	-0.081	1.502
4.	0.262	0.015	-0.050	1.529
5.	0.312	0.026	-0.013	1.563
6.	0.358	0.041	0.029	1.601
7.	0.401	0.058	0.076	1.644
8.	0.442	0.080	0.131	1.694
9.	0.482	0.105	0.198	1.754
10.	0.522	0.134	0.325	1.865
11.	0.563	0.167	0.424	1.949
12.	0.604	0.204	0.487	2.000
13.	0.647	0.246	0.539	2.040
14.	0.694	0.294	0.583	2.072
15.	0.743	0.347	0.622	2.099
16.	0.798	0.408	0.656	2.120
17.	0.861	0.479	0.686	2.138
18.	0.939	0.568	0.709	2.151
19.	1.084	0.737	0.723	2.158
20.	1.229	0.906	0.736	2.165
21.	1.305	0.996	0.757	2.175
22.	1.366	1.069	0.784	2.187
23.	1.419	1.132	0.816	2.200

Contd.....

Table 3.3 Development of Super-Conical Convolute. Example 3.4

S. No.	x_{d_i}	y_{d_i}	x_{d_j}	y_{d_j}
24.	1.465	1.188	0.852	2.214
25.	1.507	1.239	0.894	2.228
26.	1.545	1.286	0.944	2.242
27.	1.580	1.330	1.003	2.256
28.	1.613	1.370	1.075	2.270
29.	1.644	1.408	1.168	2.283
30.	1.674	1.445	1.307	2.296
31.	1.705	1.479	1.581	2.309
32.	1.736	1.514	1.734	2.308
33.	1.771	1.547	1.854	2.301
34.	1.809	1.581	1.950	2.291
35.	1.856	1.617	2.026	2.280
36.	1.915	1.656	2.081	2.270
37.	2.030	1.723	2.113	2.263
38.	2.148	1.787	2.140	2.258
39.	2.212	1.817	2.176	2.249
40.	2.267	1.837	2.216	2.238
41.	2.316	1.850	2.262	2.225
42.	2.364	1.858	2.312	2.210
43.	2.410	1.861	2.367	2.192
44.	2.456	1.858	2.431	2.171

Contd.....

Table 3.3 Development of Super-Conical Convolute Example 3.4

S. No.	x_{d_i}	y_{d_i}	x_{d_j}	y_{d_j}
45.	2.503	1.852	2.511	2.145
46.	2.551	1.840	2.661	2.094
47.	2.601	1.824	2.746	2.066
48.	2.653	1.803	2.800	2.047
49.	2.707	1.777	2.840	2.033
50.	2.765	1.745	2.873	2.021
51.	2.827	1.707	2.900	2.011
52.	2.895	1.661	2.925	2.003
53.	2.972	1.605	2.946	1.995
54.	3.065	1.533	2.963	1.989
55.	3.239	1.393	2.976	1.985
56.	3.410	1.251	2.992	1.979
57.	3.500	1.175	3.020	1.969
58.	3.570	1.111	3.059	1.954
59.	3.630	1.055	3.113	1.934
60.	3.682	1.003	3.185	1.910
61.	3.729	0.956	3.281	1.885
62.	3.771	0.914	3.406	1.867
63.	3.811	0.875	3.566	1.865
64.	3.850	0.839	3.847	1.903
65.	3.888	0.808	4.018	1.942

Contd.....

Table 3.3 Development of Super-Conical Convolute. Example 3.4

Sl. No.	x_{d_i}	y_{d_i}	x_{d_j}	y_{d_j}
66.	3.926	0.780	4.117	1.974
67.	3.965	0.755	4.191	2.004
68.	4.006	0.734	4.248	2.032
69.	4.050	0.715	4.296	2.057
70.	4.099	0.698	4.335	2.080
71.	4.156	0.685	4.367	2.099
72.	4.225	0.672	4.391	2.116
73.	4.358	0.656	4.400	2.127

Chapter 4

DEVELOPMENT OF HELICAL CONVOLUTES

4.1 Helical Convolutes

Helical convolute is the surface generated by a straight line generatrix moving in such a way that it is always tangent to a helix [2]. The convolute can be called as cylindrical helical convolute or conical helical convolute depending upon whether the helical curve lies on a cylindrical or a conical surface (refer to Figure 4.1).

4.2 Development of Helical Convolutes

Helical convolutes are tangent developable surfaces and hence satisfy the condition for developability automatically [18]. The algorithm given in Section 2.6.1 is followed to get the development. In the case of cylindrical helical convolute exact mathematical expressions have been obtained for the co-ordinates of the two ends of the generatrix at its various positions in the development of the convolute.

4.3 Conical Helical Convolutes

4.3.1 Conical Helix, the Directrix

The parametric representation of a point on the conical helix is given by

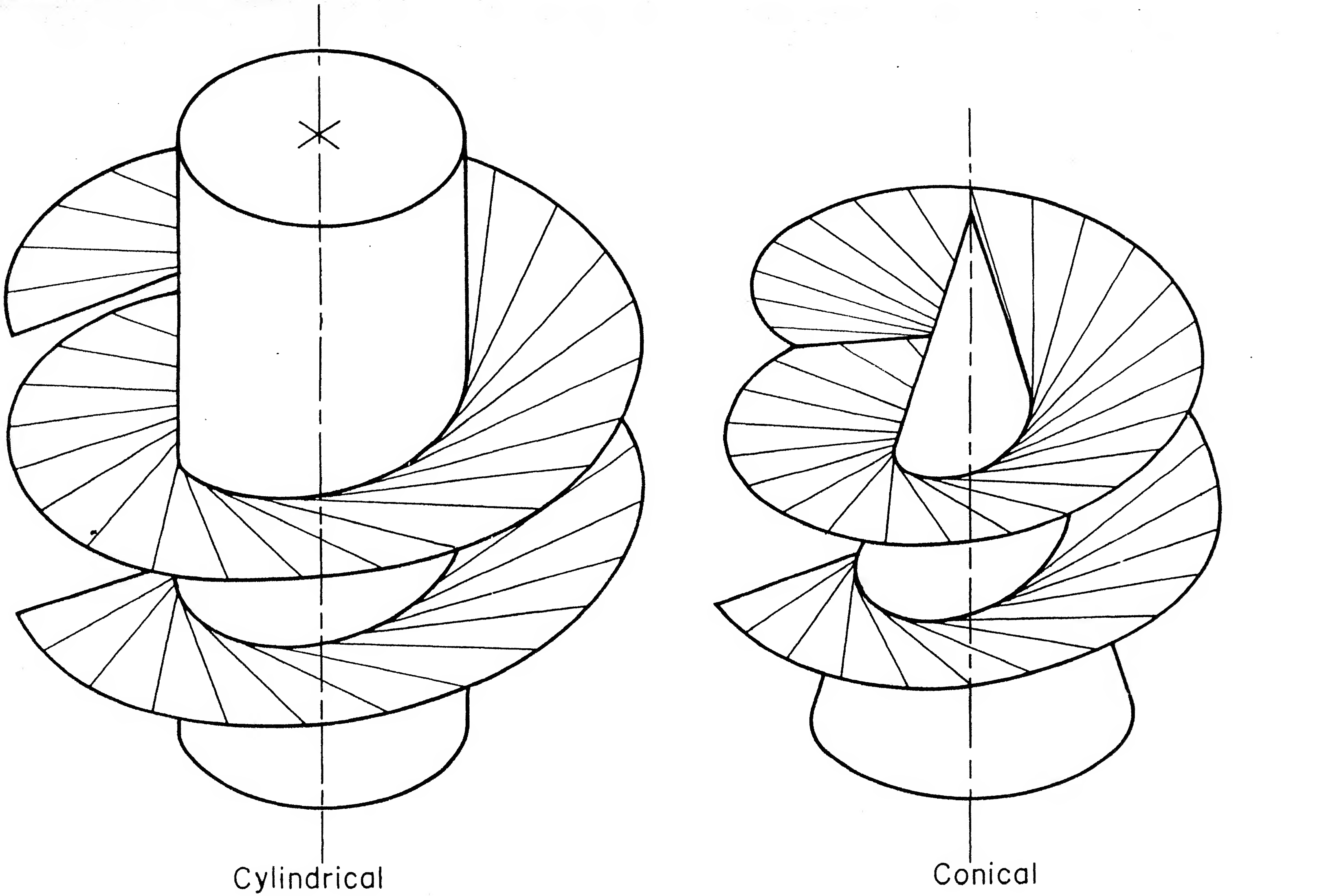


Fig. 4.1 Helical convolutes.

$$\underline{r}_i = \begin{bmatrix} a_{i_o} (1 - e_i \theta_i) \cos \theta_i \\ a_{i_o} (1 - e_i \theta_i) \sin \theta_i \\ c_i \theta_i \\ 1 \end{bmatrix} \qquad \begin{array}{l} 0 \leq \theta_i \leq 2\pi \\ a_{i_o}, e_i, c_i > 0 \end{array} \quad .... \qquad (4.1)$$

where the nomenclature is as follows:

θ_i - the parameter measured in radians,

a_{i_o} - radius of the helix at the starting point (θ_i = 0 radians),

e_i - rate of change in radius of the helix per unit radian of rotation of the generic point along the helix per unit radius at the starting point and

c_i - axial movement of the generic point along the helix per unit radian of rotation.

In general Eqn. (4.1) represents a conical helix. If $e_i = 0$, then the helix is a cylindrical one. Subscript i is used for the directrix which in the present case is a helix.

The tangent vector at any point along the helix is given by

$$\dot{\underline{r}}_i = \frac{d\underline{r}_i}{d\theta_i} = \begin{bmatrix} -a_{i_o}\{(1 - e_i\ \theta_i)\sin\theta_i + e_i\cos\theta_i\} \\ a_{i_o}\{(1 - e_i\ \theta_i)\cos\theta_i - e_i\sin\theta_i\} \\ c_i \end{bmatrix} \quad \ldots \quad (4.2)$$

The magnitude of the tangent vector is

$$\dot{s} = \left\| \dot{\underline{r}}_i \right\|$$

$$= [a_{i_o}^{\ 2}\{(1 - e_i\ \theta_i)^2 + e_i^{\ 2}\} + c_i^{\ 2}]^{1/2} \qquad (4.3)$$

The unit tangent vector is given by

$$\underline{t}_i = \frac{1}{[a_{i_o}^{\ 2}\{(1-e_i\ \theta_i)^2 + e_i^{\ 2}\} + c_i^{\ 2}]^{1/2}} \begin{bmatrix} -a_{i_o}\{(1-e_i\ \theta_i)\sin\theta_i + e_i\cos\theta_i\} \\ a_{i_o}\{(1-e_i\ \theta_i)\cos\theta_i - e_i\sin\theta_i\} \\ c_i \end{bmatrix} \quad \ldots \quad (4.4)$$

The second derivative of $\underline{r}_i$ with respect to the parameter θ_i is given by

$$\ddot{\underline{r}}_i = \frac{d^2\underline{r}_i}{d\theta_i^{\ 2}} = \begin{bmatrix} -a_{i_o}\{(1 - e_i\ \theta_i)\cos\theta_i - 2e_i\sin\theta_i\} \\ -a_{i_o}\{(1 - e_i\ \theta_i)\sin\theta_i + 2e_i\cos\theta_i\} \\ 0 \end{bmatrix} \quad \ldots \quad (4.5)$$

4.3.2 Conical Helical Convolute Surface

A general point P on the conical helical convolute surface is given by

$$\underline{r} = \underline{r}_i + \lambda\ \underline{t}_i \qquad (4.6)$$

where

$\underline{r}_i$ - the position vector of the point of tangency on the conical helix with the generatrix on which the point P lies,

$\underline{t}_i$ - the unit tangent vector to the helix at the point of tangency and

λ - distance between the point P and the point of tangency on the helix.

If L is the length of the generatrix, then the other end of it is given by

$$\underline{r}_j = \underline{r}_i + L\,\underline{t}_i$$

or,

$$\underline{r}_j = \begin{bmatrix} a_{i_o}(1-e_i\,\theta_i)\cos\theta_i - L_1\,a_{i_o}\{(1-e_i\,\theta_i)\sin\theta_i \\ \qquad + e_i\cos\theta_i\} \\ a_{i_o}(1-e_i\,\theta_i)\sin\theta_i + L_1\,a_{i_o}\{(1-e_i\,\theta_i)\cos\theta_i \\ \qquad - e_i\sin\theta_i\} \\ c_i\,\theta_i + L_1\,c_i \end{bmatrix} \qquad \ldots. \quad (4.7)$$

where

$$L_1 = \frac{L}{[a_{i_o}^{\;2}\{(1-e_i\,\theta_i)^2 + e_i^{\;2}\} + c_i^{\;2}]^{1/2}} \;.$$

4.3.3 Surface Normal

Considering the helix as a curve on the helical convolute, at any point on the helix the unit normal to the convolute surface is given by

$$\underline{n}_s = \frac{\dot{\underline{r}}_i \times (\underline{r}_j - \underline{r}_i)}{\| \dot{\underline{r}}_i \times (\underline{r}_j - \underline{r}_i) \|} \quad . \qquad (4.8)$$

But the vector $(\underline{r}_j - \underline{r}_i)$ is along the vector $\dot{\underline{r}}_i$. Hence the cross product $\dot{\underline{r}}_i \times (\underline{r}_j - \underline{r}_i)$ is a null vector. Hence the unit normal to the surface is found in another way. Since the helical convolute is generated by generatrices along the tangent to the helix, the vectors $\dot{\underline{r}}_i$ and $\ddot{\underline{r}}_i$ lie on the surface of the convolute. Hence the normal to the surface is calculated by

$$\underline{n}_s = \frac{\dot{\underline{r}}_i \times \ddot{\underline{r}}_i}{\| \dot{\underline{r}}_i \times \ddot{\underline{r}}_i \|} \quad . \qquad (4.10)$$

Substituting for $\dot{\underline{r}}_i$ and $\ddot{\underline{r}}_i$ from Eqns. (4.2) and (4.5) and simplifying,

$$\underline{n}_s = \frac{1}{\left[c_i^2 \{ (1-e_i\,\theta_i)^2 + 4\,e_i^2 \} + a_{i_o}^2 \{ (1-e_i\,\theta_i)^2 + 2\,e_i^2 \} \right]^{1/2}} \begin{bmatrix} c_i \{ (1 - e_i\,\theta_i) \sin\theta_i + 2\,e_i \cos\theta_i \} \\ -c_i \{ (1 - e_i\,\theta_i) \cos\theta_i - 2\,e_i \sin\theta_i \} \\ a_{i_o} \{ (1 - e_i\,\theta_i)^2 + 2\,e_i^2 \} \end{bmatrix}$$

... (4.11)

4.3.4 Magnitude of Geodesic Curvature

The magnitude of geodesic curvature is given by

$$k_g = \underline{n}_s \cdot \frac{\dot{\underline{r}}_i \times \ddot{\underline{r}}_i}{\dot{s}_i^{\,3}} .$$

Substituting for $\underline{n}_s$ from Eqn. (4.10) and simplifying

$$k_g = \frac{\| \dot{\underline{r}}_i \times \ddot{\underline{r}}_i \|}{\dot{s}_i^{\,3}} . \qquad (4.12)$$

When the expressions for $\dot{\underline{r}}_i$, $\ddot{\underline{r}}_i$ and $\dot{s}_i$ are substituted from Eqns. (4.2), (4.5) and (4.3) and simplified,

$$k_g = \frac{a_{i_0}\left[c_i^{\,2}\{(1-e_i\,\theta_i)^2+4\,e_i^{\,2}\}+a_{i_0}^{\,2}\{(1-e_i\,\theta_i)^2+2\,e_i^{\,2}\}^2\right]^{1/2}}{\left[a_{i_0}^{\,2}\{(1-e_i\,\theta_i)^2+e_i^{\,2}\}+c_i^{\,2}\right]^{3/2}} \qquad \ldots \quad (4.13)$$

4.3.5 Arc Length

The length of arc along the helix from the starting point to a given point is

$$s_i = \int_0^{\theta_i} \| \dot{\underline{r}}_i \| \, d\theta_i$$

$$= \int_0^{\theta_i} \left[a_{i_0}\{(1-e_i\,\theta_i)^2+e_i^{\,2}\}+c_i^{\,2}\right]^{1/2} d\theta_i \qquad \ldots . \quad (4.14)$$

4.3.6 Arc-tangent Angle

The arc-tangent angle of the helix is given by

$$\Psi_i = \int_0^{s_i} k_g\,(s_i)\,ds_i$$

where $k_g\ (s_i)$ is the geodesic curvature given as a function of the arc length. Substituting for the geodesic curvature $k_g\ (s_i)$ and ds_i, the expression for Ψ_i reduces to

$$\Psi_i = \int_0^{\theta_i} \frac{a_{i_0}\left[c_i^2 \{(1-e_i\ \theta_i)^2 + 4\ e_i^2\} + a_{i_0}^2 \{(1-e_i\ \theta_i^2) + 2\ e_i^2\}^2\right]^{1/2}}{\left[a_{i_0}^2 \{(1-e_i\ \theta_i)^2 + e_i^2\} + c_i^2\right]}\ d\theta_i \qquad \ldots \quad (4.15)$$

4.3.7 Angle Between Arc-tangent and Generatrix

Since the generatrix is along the tangent to the helix, the angle between the arc-tangent and the generatrix is zero.

4.3.8 Development of Conical Helix

The development of the primary directrix, the helix, is to be first carried out. Since the geodesic curvature, k_g, and the arc length, s_i, are continuous functions of the parameter θ_i (refer Eqns. (4.13) and (4.14)) the development of the primary directrix is carried out by integrating Serret-Frenet equations expressed in terms of the parameter θ_i (refer to Appendix II). Here, for conical helical convolutes,

$$f_1\ (\theta_i) = \frac{ds_i}{d\theta_i} = [a_{i_0}{}^2\ \{(1 - e_i\ \theta_i)^2 + e_i{}^2\} + c_i{}^2]^{1/2}$$

$$f_2\ (\theta_i) = k_g = \frac{a_{i_0}\ [c_i{}^2\ \{(1-e_i\ \theta_i)^2 + 4\ e_i{}^2\} + a_{i_0}{}^2\ \{(1-e_i\ \theta_i)^2 + 2\ e_i{}^2\}^2]^{1/2}}{[a_{i_0}{}^2\ \{(1-e_i\ \theta_i)^2 + e_i{}^2\} + c_i{}^2]^{3/2}}$$

and

$$f_3\ (\theta_i) = \frac{d\ f_1(\theta_i)}{d\theta_i} = \frac{-\ a_{i_0}{}^2\ e_i\ (1 - e_i\ \theta_i)}{[a_{i_0}{}^2\ \{(1-e_i\ \theta_i)^2 + e_i{}^2\} + c_i{}^2]^{1/2}} \qquad \ldots. \qquad (4.16)$$

Substituting Eqns. (4.16) in Eqns. (II.6) and simplifying

$$\frac{d\ Y_1}{d\theta_i} = Y_2$$

$$\frac{d\ Y_2}{d\theta_i} = \frac{-\ a_{i_0}{}^2\ e_i\ (1 - e_i\ \theta_i)}{[a_{i_0}{}^2\ \{(1 - e_i\ \theta_i)^2 + e_i{}^2\} + c_i{}^2]^{1/2}}\, Y_2 - \frac{a_{i_0}\ [c_i{}^2\ \{(1-e_i\ \theta_i{}^2) + 4\ e_i{}^2\} + a_{i_0}{}^2\ \{(1-e_i\ \theta_i{}^2) + 2\ e_i{}^2\}^2]^{1/2}}{[a_{i_0}{}^2\ \{(1 - e_i\ \theta_i)^2 + e_i{}^2\} + c_i{}^2]}\, Y_4$$

$$\frac{d\ Y_3}{d\theta_i} = Y_4$$

$$\frac{d\ Y_4}{d\theta_i} = \frac{-\ a_{i_0}{}^2\ e_i\ (1 - e_i\ \theta_i)}{[a_{i_0}{}^2\ \{(1 - e_i\ \theta_i)^2 + e_i{}^2\} + c_i{}^2]}\, Y_4 + \frac{a_{i_0}[c_i{}^2\ \{(1-e_i\ \theta_i)^2 + 4\ e_i{}^2\} + a_{i_0}{}^2\ \{(1-e_i\ \theta_i)^2 + 2\ e_i{}^2\}^2]^{1/2}}{[a_{i_0}{}^2\ \{(1 - e_i\ \theta_i)^2 + e_i{}^2\} + c_i{}^2]}\, Y_2 \qquad \ldots. \qquad (4.17)$$

where

$$Y_1 = x_{d_i}$$

$$Y_2 = \frac{dx_{d_i}}{d\theta_i} \quad (4.18)$$

$$Y_3 = y_{d_i}$$

and

$$Y_4 = \frac{dy_{d_i}}{d\theta_i} .$$

The initial conditions are

$$Y_1 = x_{d_i} = 0$$

$$Y_2 = \frac{dx_{d_i}}{d\theta_i} = f_1 (\theta_o)$$

$$Y_3 = y_{d_i} = 0$$

and

$$Y_4 = \frac{dy_{d_i}}{d\theta_i} = 0 \quad (4.19)$$

where θ_o is the initial value of θ_i. If $\theta_o = 0$, then

$$f_1 (\theta_o) = [a_{i_\theta}{}^2 (1 + e_i{}^2) + c_i{}^2]^{1/2} . \quad (4.20)$$

Integrating Eqns. (4.17) with the initial conditions given by Eqns. (4.19), the development of the primary directrix, the helix, is obtained.

4.3.9 <u>Development of Conical Helical Convolute</u> .

The development of conical helical convolute is carried out by

(1) developing the conical helix,

(2) finding the arc-tangent angle at various point along the helix and

(3) then, for the given length of the generatrix, fixing the other end of the generatrix.

For developing the helix, integration of equations (4.17) is carried out as discussed in Section 4.3.8. The arc-tangent angle, Ψ_i , and the arc length, s_i , are given by Eqns. (4.15) and (4.14) and it can be seen that

$$\Psi_i = \int_0^{\theta_i} f_1(\theta_i)\, f_2(\theta_i)\, d\theta_i$$

and

$$s_i = \int_0^{\theta_i} f_1(\theta_i)\, d\theta_i$$

(refer to Section 4.3.8).Hence the integration for Ψ_i and s_i can be carried out along with the integration of Eqns. (4.17). The initial conditions are $\Psi_i(\theta_o) = 0$ and $s_i(\theta_o) = 0$.

The algorithm for the development of the conical helical convolute is given below. Apart from developing the convolute, the position vectors of the two ends of the generatrix in the global co-ordinate frame and the geodesic curvature are found so that the orthographic views of the convolute and the graph of geodesic curvature vs. arc length can be drawn.

Step 1 : Read the values of a_{i_o} , c_i , e_i , L and ND where ND is the number of parts into which 2π radians are to be divided. The total number of points to be considered along the helix is

$$N = ND + 1$$

Step 2 : Find $\theta_{i_{INC}}$, the increment to be used for the parameter θ_i.

$$\theta_{i_{INC}} = \frac{2\pi}{ND} \text{ radians}$$

Step 3 : Set the initial conditions for integration of Eqns. (4.17), (4.15) and (4.14).

$$Y_1 (1) = 0.$$

$$Y_2 (1) = [a_{i_o}^2 (1 + e_i^2) + c_i^2]^{1/2}$$

$$Y_3 (1) = 0.$$

$$Y_4 (1) = 0.$$

$$\Psi_i (1) = 0.$$

$$s_i (1) = 0.$$

Step 4 : Set θ_i = 0 radians.

Step 5 : Do the Steps 6 through 10 for K=1 to N.

Step 6 : If K=1, go to Step 7. If K>1, go to Step 8.

Step 7 : Calculate, at the initial position, the position vector of the two ends of the generatrix (Eqns. (4.1) and (4.7)) and also the geodesic curvature

(Eqn. (4.13)).

$$\underline{r}_i(1) = \begin{bmatrix} a_{i_O} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\underline{r}_j(1) = \begin{bmatrix} a_{i_O} - L_1\ (1)\ a_{i_O} e_i \\ L_1\ (1)\ a_{i_O} \\ L_1\ (1)\ b_i \\ 1 \end{bmatrix}$$

and

$$k_g(1) = \frac{a_{i_O}[c_i^2(1 + 4\ e_i^2) + a_{i_O}^2(1 + 2\ e_i^2)^2]^{1/2}}{[a_{i_O}^2\ (1 + e_i^2) + c_i^2]^{3/2}}$$

where $$L_1(1) = \frac{L}{[a_{i_O}^2\ (1 + e_i^2) + c_i^2]^{1/2}}$$

Go to Step 10.

Step 8 : Find the values of $f_1\ (\theta_i)$, $f_2\ (\theta_i)$ and $f_3\ (\theta_i)$ from Eqns. (4.16). Carry out the integration of Eqns. (4.17), (4.15) and (4.14) over the range θ_{i_p} to θ_i ($\theta_{i_p} = \theta_i - \theta_{i_{INC}}$) to find the values of Y_1 (K), Y_2 (K), Y_3 (K), Y_4 (K), Ψ_i (K) and s_i (K). Also set

$$k_g\ (K) = f_2\ (\theta_i).$$

Step 9 : Find the position vector of the two ends of the generatrix, $\underline{r}_i$ (K) and $\underline{r}_j$ (K), from Eqns. (4.2) and (4.7).

Step 10: Find the co-ordinates of the two ends of the generatrix in the development, given by

x_{d_i} (K) = Y_1 (K) ,

y_{d_i} (K) = Y_3 (K) ,

x_{d_j} (K) = x_{d_i} (K) + L cos Ψ_i (K) and

y_{d_j} (K) = y_{d_i} (K) + L sin Ψ_i (K);

x_{d_i} and y_{d_i} are the co-ordinates of the end of the generatrix lying on the helix; x_{d_j} and y_{d_j} are the co-ordinates of the other end of the generatrix.

Step 11: Set $\theta_{i_p} = \theta_i$; $\theta_i = \theta_i + \theta_{i_{INC}}$. Go to Step 8.

4.4 Development of Cylindrical Helical Convolute

For cylindrical helical convolute the various expressions given in Section 4.3 are reduced by putting $c_i = 0$.

$$\underline{r}_i = \begin{bmatrix} a_{i_0} \cos \theta_i \\ a_{i_0} \sin \theta_i \\ c_i \theta_i \\ 1 \end{bmatrix} \qquad (4.1a)$$

$$\dot{\underline{r}}_i = \begin{bmatrix} -a_{i_0} \sin \theta_i \\ a_{i_0} \cos \theta_i \\ c_i \end{bmatrix} \tag{4.2a}$$

$$\dot{s}_i = (a_{i_0}{}^2 + c_i{}^2)^{1/2} \tag{4.3a}$$

$$\underline{t}_i = \frac{1}{(a_{i_0}{}^2 + c_i{}^2)^{1/2}} \begin{bmatrix} -a_{i_0} \sin \theta_i \\ a_{i_0} \cos \theta_i \\ c_i \end{bmatrix} \tag{4.4a}$$

$$\ddot{\underline{r}}_i = \begin{bmatrix} -a_{i_0} \cos \theta_i \\ -a_{i_0} \sin \theta_i \\ 0 \end{bmatrix} \tag{4.5a}$$

$$\underline{r}_j = \begin{bmatrix} a_{i_0} (\cos \theta_i - L_1 \sin \theta_i) \\ a_{i_0} (\sin \theta_i + L_1 \cos \theta_i) \\ c_i (\theta_i + L_1) \\ 1 \end{bmatrix} \tag{4.7a}$$

where $$L_1 = \frac{1}{(a_{i_0}{}^2 + c_i{}^2)^{1/2}} .$$

$$k_g = \frac{a_{i_0}}{(a_{i_0}{}^2 + c_i{}^2)} \tag{4.13a}$$

$$s_i = (a_{i_0}{}^2 + c_i{}^2)^{1/2} \theta_i \tag{4.14a}$$

$$\Psi_i = \frac{a_{i_o}}{(a_{i_o}^2 + c_i^2)^{1/2}} \theta_i \tag{4.15a}$$

From the above equations it can be seen that the arc length s_i and the arc-tangent Ψ_i are proportional to θ_i. Also it can be seen that the geodesic curvature k_g is independent of θ_i. It is a function of a_{i_o} and b_i only and hence a constant for a given cylindrical helical convolute. It is independent of arc length.

The integration of Serret-Frenet equations has been done and expressions for the co-ordinates of the end points of the generatrix in the development have been obtained. They are :

$$x_{d_i} = \left(\frac{a_{i_o}^2 + c_i^2}{a_{i_o}}\right) \sin\left\{\frac{a_{i_o}}{(a_{i_o}^2 + c_i^2)^{1/2}}\right\} \theta_i$$

$$y_{d_i} = \left(\frac{a_{i_o}^2 + c_i^2}{a_{i_o}}\right) \left[1 - \cos\left\{\frac{a_{i_o}}{(a_{i_o}^2 + c_i^2)^{1/2}}\right\}\theta_i\right]$$

$$x_{d_j} = x_{d_i} + L \cos\left\{\frac{a_{i_o}}{(a_{i_o}^2 + c_i^2)^{1/2}}\right\} \theta_i \tag{4.21}$$

$$y_{d_j} = y_{d_i} + L \sin\left\{\frac{a_{i_o}}{(a_{i_o}^2 + c_i^2)^{1/2}}\right\} \theta_i \ .$$

4.4.1 <u>Algorithm for the Development of Cylindrical Helical Convolute</u>

In the algorithm given below, apart from finding the details of the development of cylindrical helical

convolute, details of the convolute to draw its orthographic views are also found. Also the magnitude of geodesic curvature, arc length and arc-tangent angle are found.

Step 1 : Read the values of a_{i_O}, c_i, L and ND where ND is the number of parts into which 2π radians are to be divided. The total number of points to be considered along the helix is

$$N = ND + 1$$

Step 2 : Find $\theta_{i_{INC}}$, the increment to be used for the parameter θ_i.

$$\theta_{i_{INC}} = \frac{2\pi}{ND} \text{ radians.}$$

Step 3 : Calculate the value of L_1.

Step 4 : Calculate the magnitude of geodesic curvature from Eqn. (4.13a).

Step 5 : Set $\theta_i = 0$ radians.

Step 6 : Do the following steps for K = 1 to N.

Step 7 : Find the position vector of the two ends of the generatrix, $\underline{r}_i$ (K) and $\underline{r}_j$ (K) from Eqns. (4.2a) and (4.7a).

Step 8 : Find the arc length, s_i (K), and the arc-tangent angle, Ψ_i (K), from Eqns. (4.14a) and (4.15a) respectively.

Step 9 : Find the co-ordinates of the two ends of the generatrix in the development, x_{d_i} (K), y_{d_i} (K), x_{d_j} (K) and y_{d_j} (K), from Eqns. (4.21).

Step 10: Set $\theta_i = \theta_i + \theta_{i_{INC}}$. Go to Step 6.

4.4.2 Drawing of the Development

From Eqns. (4.21) it can be seen that the development of the cylindrical helical convolute consists of two concentric circular arcs AB and CD and two straight lines AC and BD (refer to Figure 4.2). The straight lines AC and BD are tangent to the arc AB at A and B respectively. The co-ordinates of O, the centre of the arcs AB and CD, is

$$(0, \frac{a_{i_0}^2 + c_i^2}{a_{i_0}}) .$$

The radius of the arc AB, which is the development of the directrix, viz. the cylindrical helix, is

$$r_p = \frac{a_{i_0}^2 + c_i^2}{a_{i_0}} . \tag{4.22}$$

The radius of the arc CD, which is the development of the curve generated by the other end of the generatrix (this is also a cylindrical helix), is

$$r_s = [(\frac{a_{i_0}^2 + c_i^2}{a_{i_0}})^2 + L^2]^{1/2} \tag{4.23}$$

The angle of both the arcs is $\dfrac{a_{i_0}}{(a_{i_0}^2+c_i^2)^{1/2}}\ 2\pi$ radians.

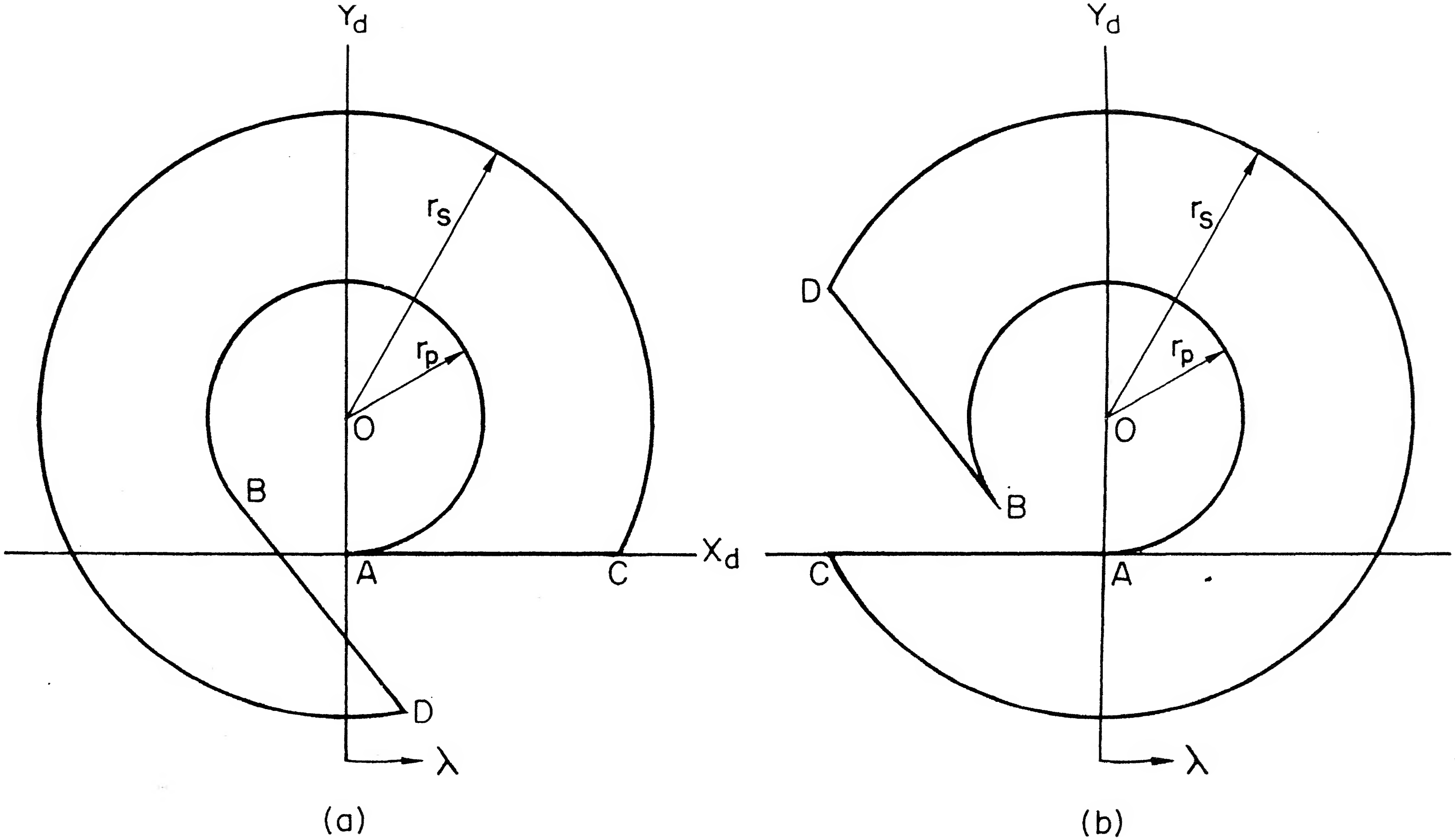


Fig. 4.2 Development of cylindrical helical convolute.

Measuring the angle λ from the line OA, the angular positions of lines OA, OB, OC and OD are given by

$$\lambda_A = 0$$

$$\lambda_B = \frac{a_{i_O}}{(a_{i_O}{}^2+c_i{}^2)^{1/2}} 2\pi \qquad (4.24)$$

$$\lambda_C = \tan^{-1} \left(\frac{a_{i_O} L}{a_{i_O}{}^2 + c_i{}^2}\right)$$

$$\lambda_D = \lambda_C + \lambda_C$$

radians respectively.

If the generatrix is in the positive direction of the tangent to the helix, then the value of L is positive. So the angle λ_C is also positive. The development of the cylindrical helical convolute will be as shown in Figure 4.2a. If the generatrix is in the negative direction to the tangent to the helix, then L and hence λ_C are negative and the development of the convolute will be as shown in Figure 4.2b.

4.5 Case Studies

The methods for the development of helical convolutes discussed above are illustrated by two examples, one for conical helical convolute and the other for cylindrical helical convolute. Based on the algorithms given in Sections 4.3.9 and 4.4.1, computer programmes for the

development of conical helical convolutes and cylindrical helical convolutes have been developed and used. These programmes take as input data the details of the helix and the length of the generatrix. Details of the helical convolute and its development are calculated for one complete rotation along the helix and are presented in tabular statements. Also graphic module is available so that the following figures can be seen on the computer graphics terminal:

(a) orthographic views of the helical convolute

(b) development of the helical convolute

(c) graph of magnitude of geodesic curvature vs. arc length of the helix (in the case of conical helical convolutes).

4.5.1 Example 4.1 : Conical Helical Convolute

Following is the input data:

(a) radius of the conical helix at the starting point, a_{i_o} = 5.0

(b) axial movement of the generic point along the helix per unit radian of rotation, c_i = 2.0

(c) rate of change in radius of the helix per unit radian of rotation of the generic point along the helix per unit radius at the starting point, e_i = 0.1

(d) length of generatrix, L = 8.0

(e) number of parts into which one complete rotation is divided, ND = 72

The output data is presented in graphical form in Figure 4.3.

4.5.2 Example 4.2: Cylindrical Helical Convolute

The input data is:

(a) radius of cylindrical helix, a_{i_0} = 2.0

(b) axial movement of the generic point along the helix per unit radian of rotation, c_i = 1.2

(c) length of generatrix, L = 5.0

(d) number of parts into which one complete rotation is divided, ND = 72

The output data is graphically presented in Figure 4.4. Since the magnitude of the geodesic curvature is constant throughout the helix, the graph of magnitude of geodesic curvature vs. arc length is not given.

4.5.3 Example 4.3 : Conical Helical Convolute with Elliptical Base

So far helices having circular base have been considered. The base can also be an ellipse. The surface obtained by having the generatrices along the tangent to such a helix is conical helical convolute with elliptical

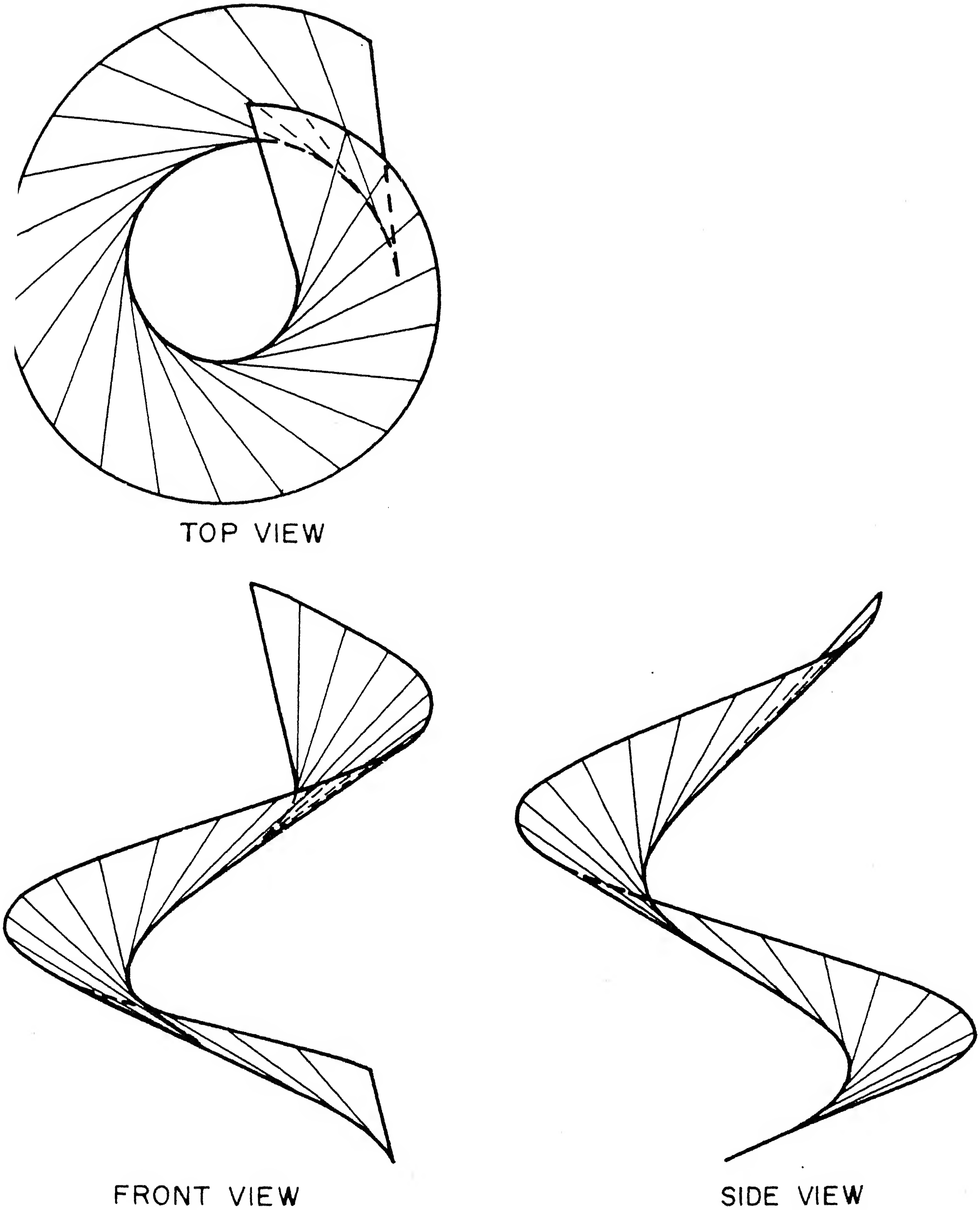


Fig. 4.3 a Orthographic views of conical helical convolute. Example 4.1.

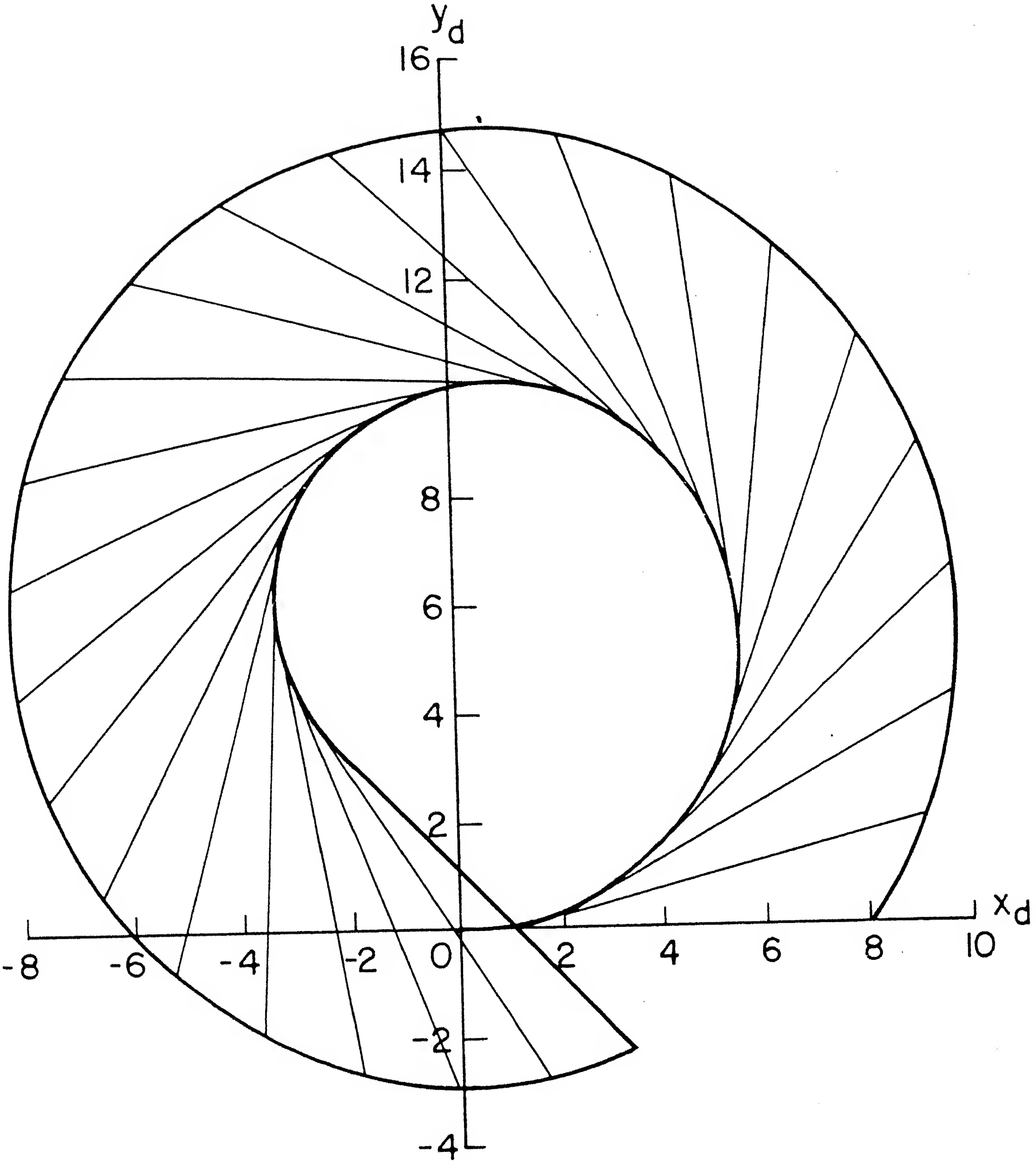


Fig. 4.3 b Development of conical helical convolute. Example 4.1

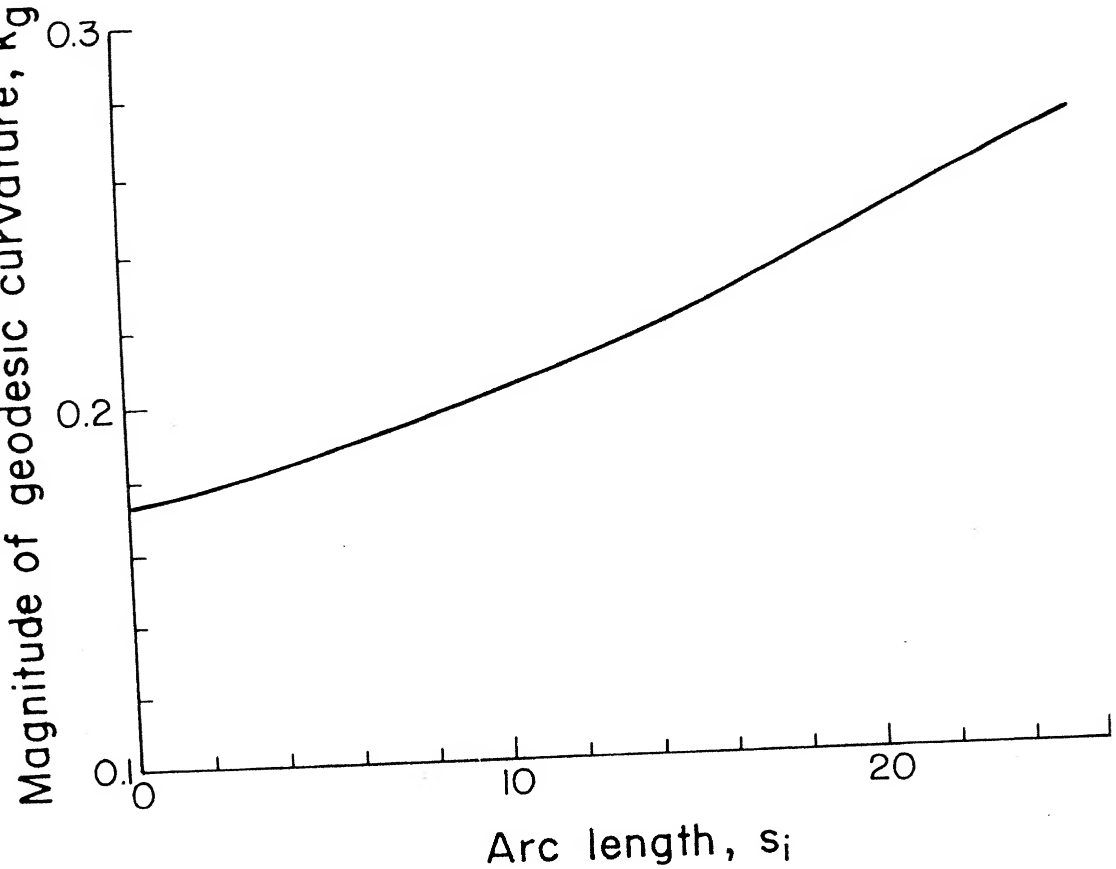


Fig. 4.3c Magnitude of geodesic curvature vs. arc length. Example 4.1

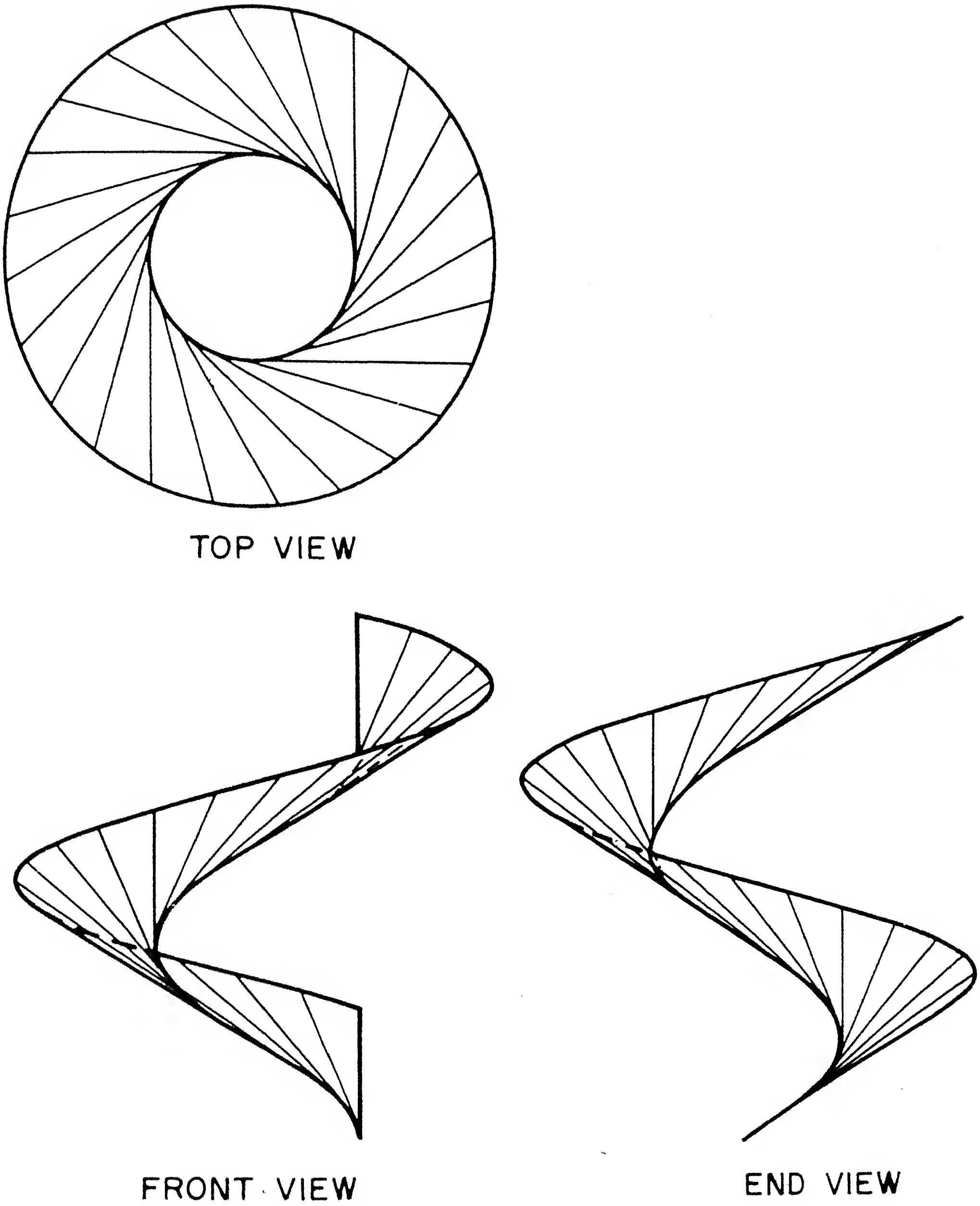


Fig.4.4a Orthographic views of cylindrical helical convolute. Example 4.2

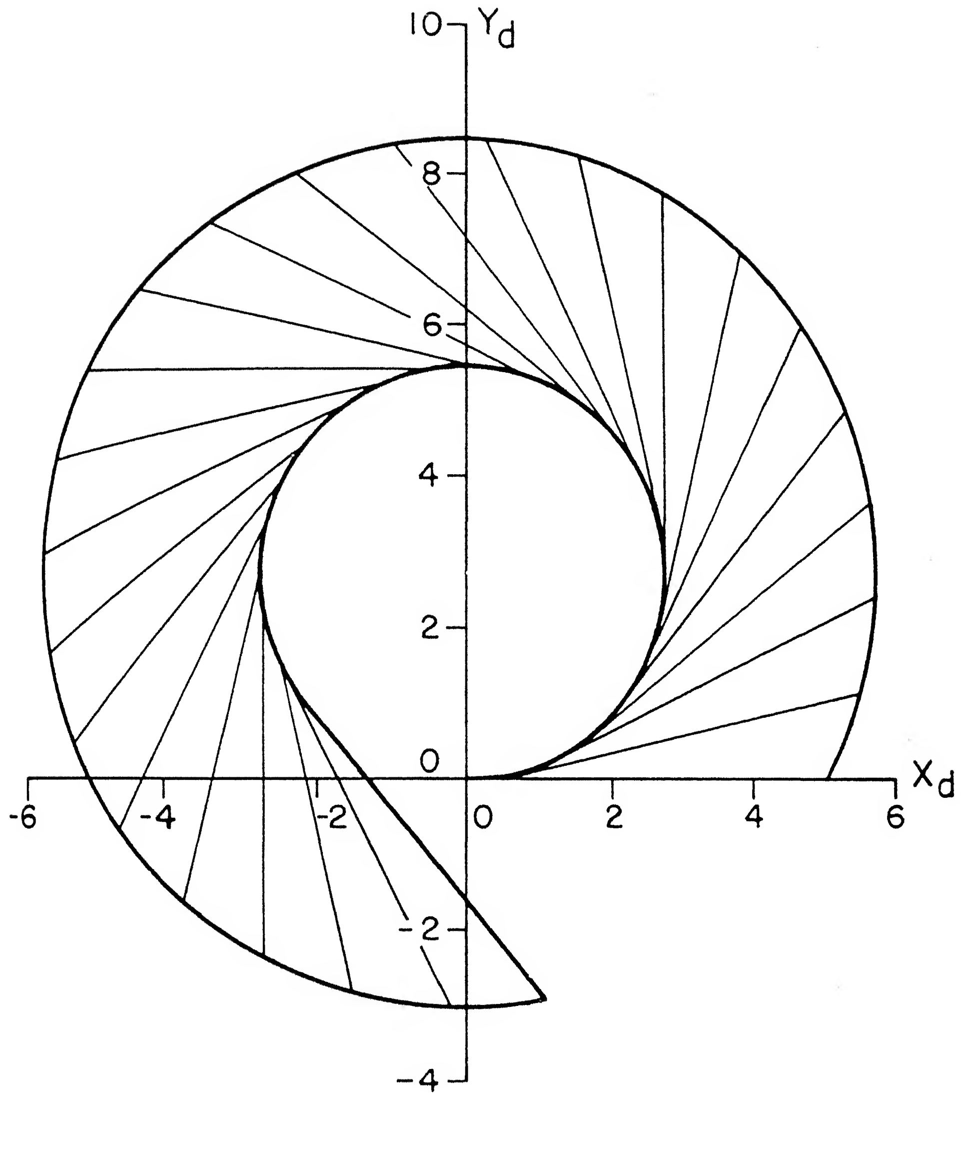


Fig. 4.4b Development of cylindrical helical convolute.
Example 4.2.

base. The various mathematical expressions of interest are given below. If

(i) a_{i_O} is the semi-major diameter of the ellipse at the starting,

(ii) b_{i_O} is the semi-minor diameter of the ellipse at the starting,

(iii) c_i is the axial movement of the generic point of the helix per unit radian rotation and

(iv) e_i is the rate at which the semi-major and semi-minor diameters change per unit radian rotation per unit value of them at the starting point

then,

$$\underline{r}_i = \begin{bmatrix} a_{i_O} (1 - e_i \theta_i) \cos \theta_i \\ b_{i_O} (1 - e_i \theta_i) \sin \theta_i \\ c_i \theta_i \\ 1 \end{bmatrix} \qquad (4.25)$$

$$\dot{\underline{r}}_i = \begin{bmatrix} -a_{i_O} \{(1 - e_i \theta_i) \sin \theta_i + e_i \cos \theta_i\} \\ b_{i_O} \{(1 - e_i \theta_i) \cos \theta_i - e_i \sin \theta_i\} \\ c_i \end{bmatrix} \qquad \ldots \quad (4.26)$$

$$\dot{s}_i = [a_{i_O}{}^2 \{(1 - e_i \theta_i) \sin \theta_i + e_i \cos \theta_i\}^2 + b_{i_O}{}^2 \{(1 - e_i \theta_i) \cos \theta_i - e_i \sin \theta_i\}^2 + c_i{}^2]^{1/2} \qquad \ldots \quad (4.27)$$

$$\underline{t} = \frac{1}{(\text{Factor } 1)_i} \begin{bmatrix} -a_{i_O} \{ (1 - e_i\, \theta_i) \sin \theta_i + e_i \cos \theta_i \} \\ b_{i_O} \{ (1 - e_i\, \theta_i) \cos \theta_i - e_i \sin \theta_i \} \\ c_i \end{bmatrix}$$

where

$$(\text{Factor } 1)_i = \dot{s}_i \qquad \ldots \qquad (4.28)$$

$$\underline{r}_j = \begin{bmatrix} a_{i_O} \; (1 - e_i\, \theta_i) \cos \theta_i \\ b_{i_O} \; (1 - e_i\, \theta_i) \sin \theta_i \\ c_i\, \theta_i \\ 1 \end{bmatrix}$$

$$+ \frac{L}{(\text{Factor } 1)_i} \begin{bmatrix} -a_{i_O} \{(1 - e_i\, \theta_i) \sin \theta_i + e_i \cos \theta_i \} \\ b_{i_O} \{(1 - e_i\, \theta_i) \cos \theta_i - e_i \sin \theta_i \} \\ c_i \\ 0 \end{bmatrix}$$

$$\ldots \qquad (4.29)$$

$$\ddot{\underline{r}}_i = \begin{bmatrix} - a_{i_O} \{ (1 - e_i\, \theta_i) \cos \theta_i - 2\, e_i \sin \theta_i \} \\ - b_{i_O} \{(1 - e_i\, \theta_i) \sin \theta_i + 2\, e_i \cos \theta_i \} \\ 0 \end{bmatrix} \qquad (4.30)$$

$$k_g = \frac{\left[b_{i_O}^{\,2}\, c_i^{\,2} \{(1 - e_i\, \theta_i) \sin \theta_i + 2\, e_i \cos \theta_i \}^2 + a_{i_O}^{\,2}\, c_i^{\,2} \{ (1 - e_i\, \theta_i) \cos \theta_i - 2\, e_i \sin \theta_i \}^2 + a_{i_O}^{\,2}\, b_{i_O}^{\,2} \{(1 - e_i\, \theta_i)^2 + 2\, e_i^{\,2} \}^2 \right]^{1/2}}{\left[a_{i_O}^{\,2} \{ (1 - e_i\, \theta_i) \sin \theta_i + e_i \cos \theta_i \}^2 + b_{i_O}^{\,2} \{(1 - e_i\, \theta_i) \cos \theta_i - e_i \sin \theta_i \}^2 + c_i^{\,2} \right]^{3/2}}$$

$$\ldots \qquad (4.31)$$

$$s_i = \int_0^{\theta_i} [a_{i_0}{}^2 \{(1 - e_i\, \theta_i) \sin \theta_i + e_i \cos \theta_i\}^2$$
$$+ b_{i_0}{}^2 \{(1 - e_i\, \theta_i) \cos \theta_i - e_i \sin \theta_i\}^2$$
$$+ c_i{}^2]^{1/2}\, d\theta_i \qquad (4.32)$$

The equations to be considered for integration of Serret-Frenet Equations are (refer Appendix III and Section 4.3.8):

$$\frac{d\, Y_1}{d\theta_i} = Y_2$$
$$\frac{d\, Y_2}{d\theta_i} = \frac{g_1\, (\theta_i)}{g_2\, (\theta_i)}\, Y_2 - \frac{g_3(\theta_i)}{g_2(\theta_i)}\, Y_4$$
$$\frac{d\, Y_3}{d\theta_i} = Y_4 \qquad (4.33)$$
$$\frac{d\, Y_4}{d\theta_i} = \frac{g_1\, (\theta_i)}{g_2\, (\theta_i)}\, Y_4 + \frac{g_3\, (\theta_i)}{g_2\, (\theta_i)}\, Y_2$$

where

$$g_1\, (\theta_i) = (a_{i_0}{}^2 - b_{i_0}{}^2) \{(1 - e_i\, \theta_i)^2 - 2\, e_i\} \sin \theta_i \cos \theta_i$$
$$+ e_i\, (1 - e_i\, \theta_i) \{ (a_{i_0}{}^2 - 2\, b_{i_0}{}^2) \cos^2\theta_i$$
$$+ (b_{i_0}{}^2 - 2\, a_{i_0}{}^2) \sin^2\theta_i \},$$

$$g_2\, (\theta_i) = a_{i_0}{}^2 \{(1 - e_i\, \theta_i) \sin \theta_i + e_i \cos \theta_i\}^2$$
$$+ b_{i_0}{}^2 \{(1 - e_i\, \theta_i) \cos \theta_i - e_i \sin \theta_i\}^2$$
$$+ c_i{}^2 \quad \text{and}$$

$$g_3\, (\theta_i) = [b_{i_0}{}^2\, c_i{}^2 \{(1 - e_i\, \theta_i) \sin \theta_i + 2\, e_i \cos \theta_i\}$$
$$+ a_{i_0}{}^2\, c_i{}^2 \{(1 - e_i\, \theta_i) \cos \theta_i - 2\, e_i \sin \theta_i\}$$
$$+ a_{i_0}{}^2\, b_{i_0}{}^2 \{ (1 - e_i\, \theta_i)^2 + 2\, e_i{}^2\}^2]^{1/2} \quad .. \qquad (4.34)$$

The initial conditions are

$Y_1 = 0$,

$$Y_2 = [a_{i_o}^{\,2}\{(1 - e_i\,\theta_o)\sin\theta_o + e_i\cos\theta_o\}^2 + b_{i_o}^{\,2}\{(1 - e_i\,\theta_o)\cos\theta_o - e_i\sin\theta_o\}^2 + c_i^{\,2}]^{1/2},$$

$Y_3 = 0$ and

$$Y_4 = 0 \tag{4.35}$$

where θ_o is the initial value of θ_i.

The algorithm for the development of the conical helical convolute with elliptical base is similar to the algorithm for the conical helical convolute given in Section 4.3.9. An illustrative example showing how the development of a conical helical convolute with elliptical base can be obtained is given here.

Following is the input data:

(a) Semi-major diameter at the starting point, a_{i_o} = 5.0

(b) Semi-minor diameter at the starting point, b_{i_o} = 3.0

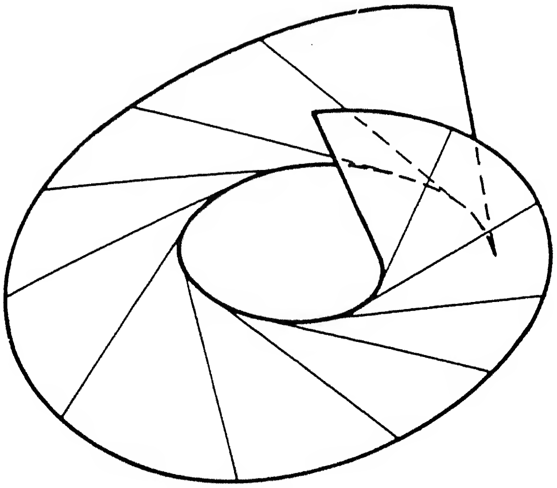
(c) Axial movement of generic point of helix per unit radian rotation, c_i = 2.0

(d) Rate at which semi-major and semi-minor diameters reduce per unit radian rotation per unit value of them at the starting point, e_i = 0.1

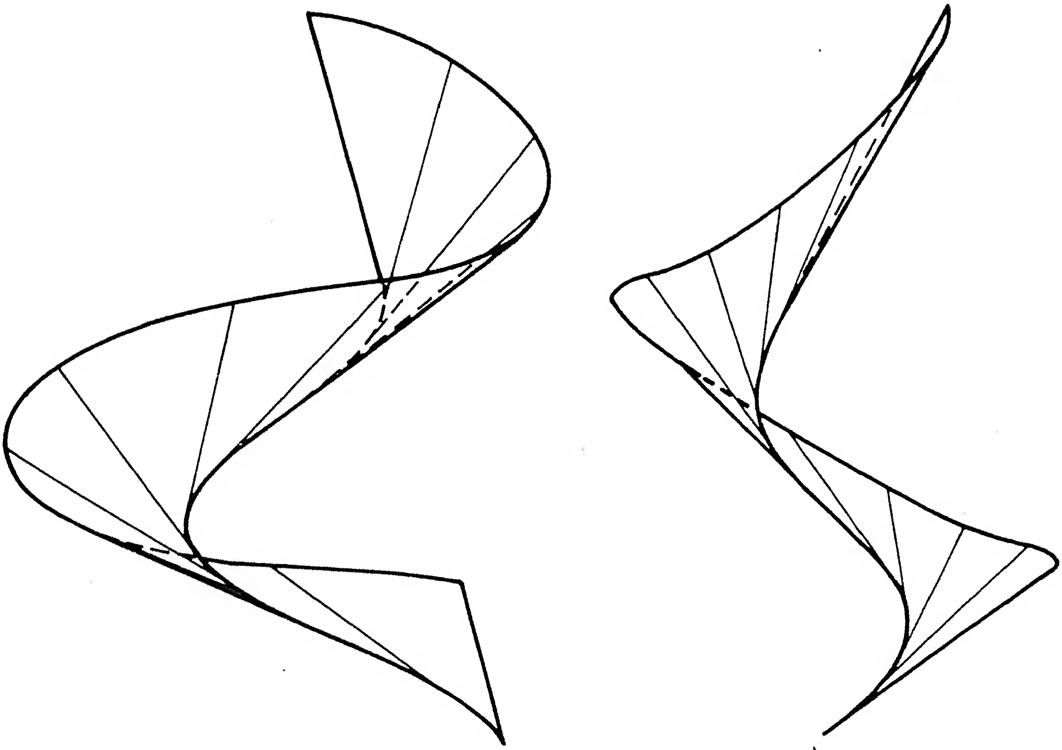
(e) Length of the generatrix, L = 8.0

(f) Number of parts into which one complete rotation is divided, ND =72

The output data is presented in graphical form in Figure 4.5.



TOP VIEW



FRONT VIEW SIDE VIEW

Fig. 4.5a Orthographic views of conical helical convolute with elliptical base. Example 4.3

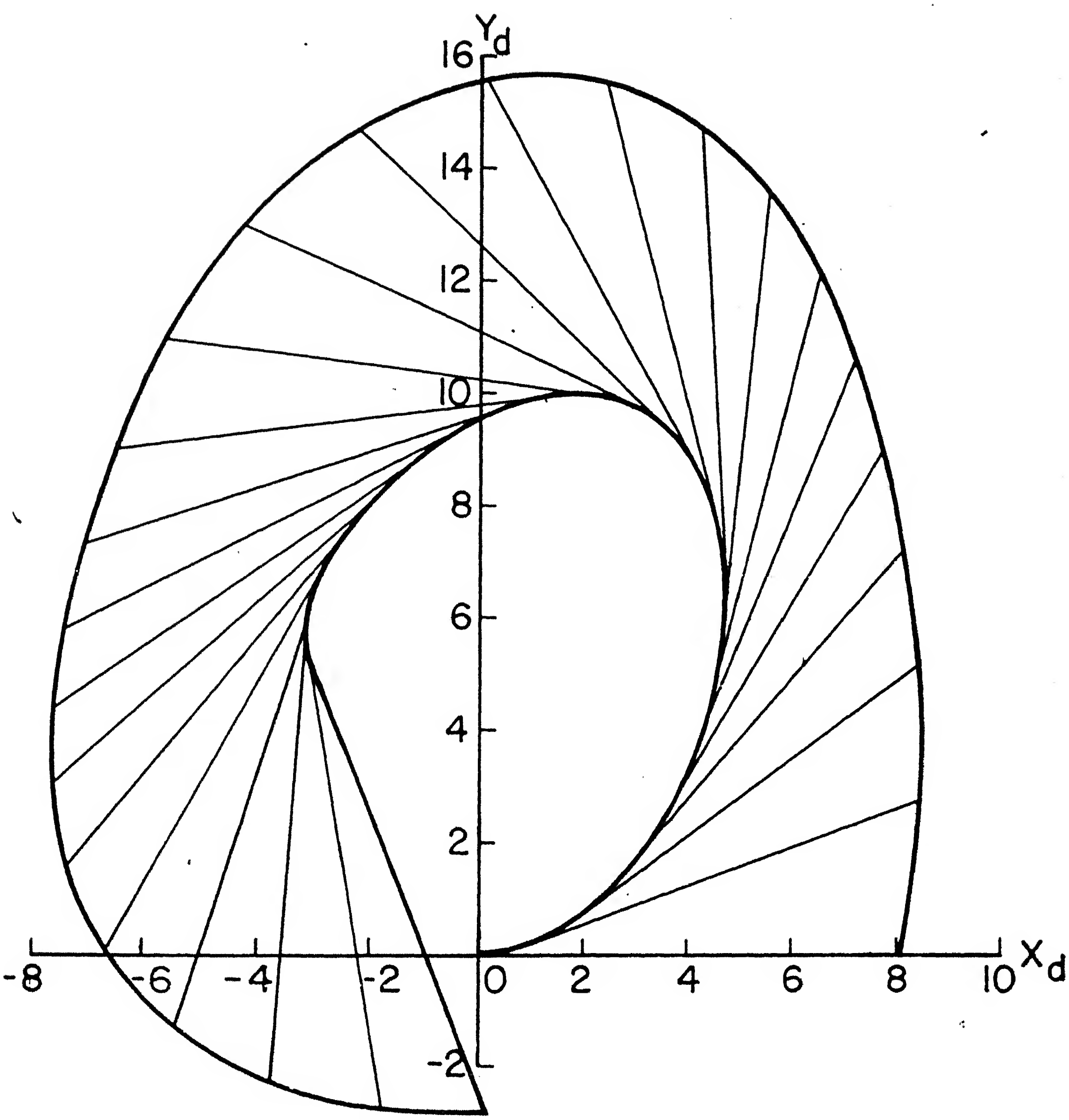


Fig. 4.5b Development of conical helical convolute with elliptical base. Example 4.3

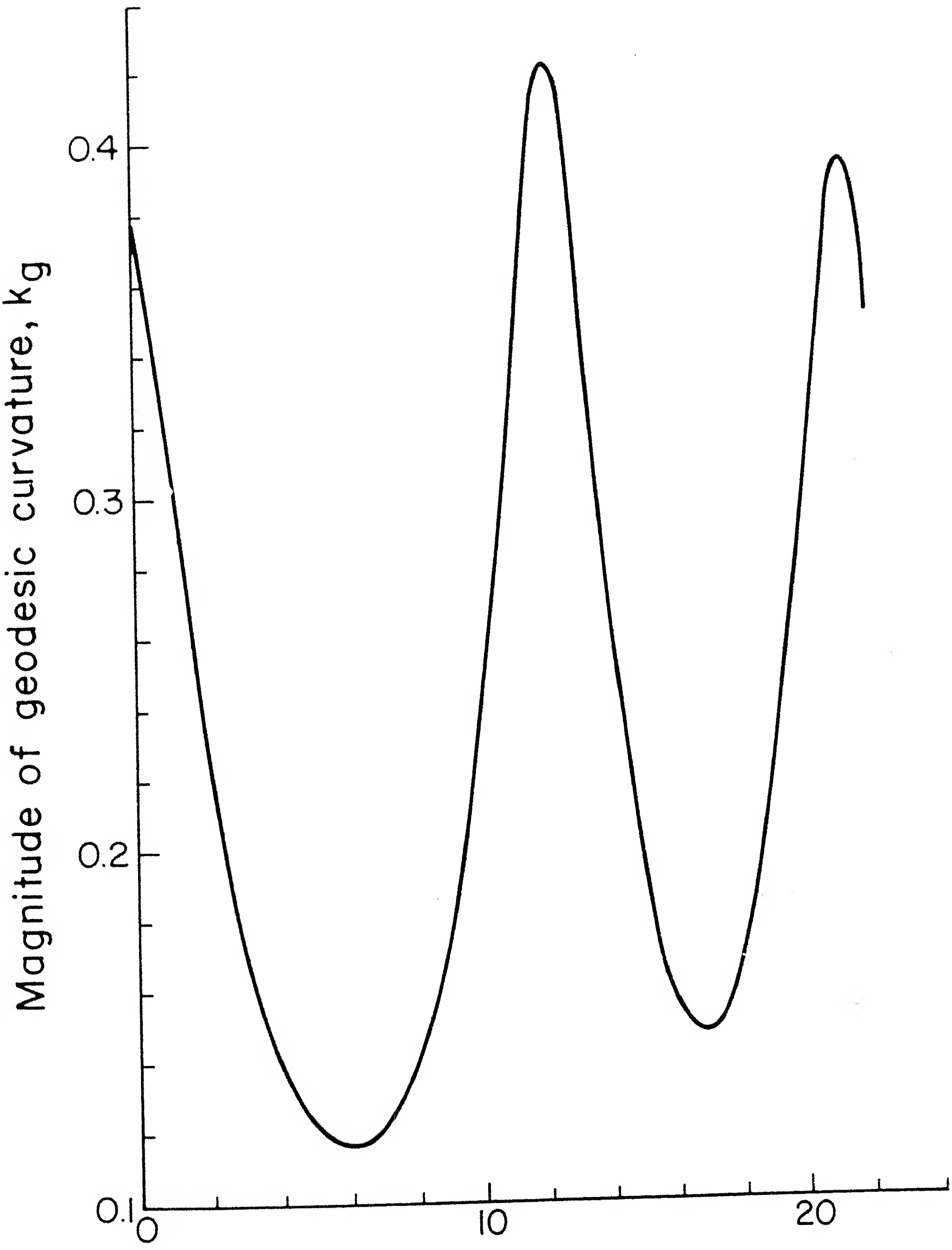


Fig. 4.5c Magnitude of geodesic curvature vs. arc length. Example 4.3.

Chapter 5

DEVELOPMENT OF THIN DUCTS

5.1 Introduction

Ducts are closed tubular surfaces with open throats or ends such that the dimension along the centre line of the duct is large compared to the dimensions perpendicular to the centre line. Development of thin ducts (i.e. ducts in which thickness of surface is negligible) is given in this chapter.

5.2 Geometry of Ducts

The following sections, Section 5.2.1 to 5.2.4, indicate how the entire geometry of a duct can be expressed in terms of a single parameter.

5.2.1 Centre Line of the Duct

The centre line of a duct is a space curve or a planar curve as the case may be. If it is a planar curve, then the duct is called a planar duct; otherwise it is called a duct in space. The centre line, being a curve, is governed by a single parameter and is defined by (refer to Figure 5.1)

$$\underline{r}_a = \begin{bmatrix} x_a\ (\theta_a) \\ y_a\ (\theta_a) \\ z_a\ (\theta_a) \\ 1 \end{bmatrix} \qquad \theta_{a_s} \leq \theta_a \leq \theta_{a_f} \tag{5.1}$$

where

$\underline{r}_a$ - position vector of a generic point of the centre line of the duct in global co-ordinates,

θ_a - parameter of the centre line,

θ_{a_s} - value of the parameter at the starting point on the centre line,

θ_{a_f} - value of the parameter at the end point on the centre line and

$\begin{bmatrix} x_a\ (\theta_a) \\ y_a\ (\theta_a) \\ z_a\ (\theta_a) \end{bmatrix}$ - vector function defining the centre line of the duct in terms of the parameter θ_a.

Here only those curves that are smooth and continuous are considered. It is assumed that singular points or discontinuities do not exist along the center line.

5.2.2 Cross Section of the Duct : Plane of Cross Section

Corresponding to any point along the centre line of the duct, the cross section of the duct is defined by the intersection of the normal plane of the

centre line at that point with the surface of the duct. The tangent to the centre line at that point is normal to the plane of cross section of the duct. A local co-ordinate frame is defined for the cross section of the duct such that the origin is at the point on the centre line and the X, Y and Z axes are along the normal, bi-normal and tangent vectors respectively to the centre line of the duct at that point. Then a generic point of the curve defining the cross section of the duct is given by

$$\underline{r} = \begin{bmatrix} n_{a_x} & b_{a_x} & t_{a_x} & r_{a_x} \\ n_{a_y} & b_{a_y} & t_{a_y} & r_{a_y} \\ n_{a_z} & b_{a_z} & t_{a_z} & r_{a_z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \underline{r}_L \tag{5.2}$$

where

$\underline{r}$ - position vector, in global co-ordinates, of the generic point,

$\underline{r}_L$ - position vector, in local co-ordinates, of the generic point

$\underline{n}_a$, $\underline{b}_a$, $\underline{t}_a$ - normal, bi-normal and tangent vectors to the centre line of the duct and

$\underline{r}_a$ - position vector of the point on the centre line of the duct.

The suffices x, y and z indicate the components along the X, Y and Z axes in the global co-ordinate frame. The vectors $\underline{n}_a$, $\underline{b}_a$ and $\underline{t}_a$ are obviously functions of θ_a. Expressions for $\underline{n}_a$, $\underline{b}_a$ and $\underline{t}_a$ can be derived using Eq. (5.1) [18] and are as follows:

$$\underline{t}_a = \frac{\dot{\underline{r}}_a}{(\dot{\underline{r}}_a \cdot \dot{\underline{r}}_a)^{1/2}} ,$$

$$k\underline{n}_a = \frac{(\dot{\underline{r}}_a \cdot \dot{\underline{r}}_a)\,\ddot{\underline{r}}_a - (\dot{\underline{r}}_a \cdot \ddot{\underline{r}}_a)\,\dot{\underline{r}}_a}{(\dot{\underline{r}}_a \cdot \dot{\underline{r}}_a)^2} ;$$

$$k^2 = \frac{(\dot{\underline{r}}_a \times \ddot{\underline{r}}_a) \cdot (\dot{\underline{r}}_a \times \ddot{\underline{r}}_a)}{(\dot{\underline{r}}_a \cdot \dot{\underline{r}}_a)^3}$$

$$\underline{b}_a = \underline{t}_a \times \underline{n}_a$$

where

$$\dot{\underline{r}}_a = \frac{d\underline{r}_a}{d\theta_a} \quad \text{and} \quad \ddot{\underline{r}}_a = \frac{d^2\underline{r}_a}{d\theta_a^{\,2}}$$

5.2.3 Cross Section of the Duct : Shape and Size

The cross section of the duct can vary in shape and size along the length of the duct. This variation can be taken care of by expressing the details governing the shape and size of the cross section of the duct as functions of the parameter θ_a. The curve defining the cross section is a one parameter vector function. Let θ_{cs} be the parameter. Apart from the function, the

constants in the vector function control the shape and size of the curve. If these constants are made to be functions of the parameter θ_a, then depending upon the value of θ_a, the shape and size of the cross section of the duct will vary. A generic point of the curve of cross section is then given by

$$\underline{r}_L = \begin{bmatrix} x_{cs}(\theta_a, \theta_{cs}) \\ y_{cs}(\theta_a, \theta_{cs}) \\ z_{cs}(\theta_a, \theta_{cs}) \\ 1 \end{bmatrix} \qquad (5.3)$$

where x_{cs}, y_{cs} and z_{cs} are components of the vector function $\underline{r}_L$ along the X, Y and Z axes of the local co-ordinate frame. Generally, the curve of cross section will be a planar curve lying in the XY plane of the local co-ordinate system. In such a case, the z_{cs} component in Eq. (5.3) will be zero.

5.2.4 Cross Section of the Duct

From Eqns. (5.2) and (5.3), the generic point of the curve of cross section is given by

$$\underline{r} = \begin{bmatrix} n_{a_x} & b_{a_x} & t_{a_x} & r_{a_x} \\ n_{a_y} & b_{a_y} & t_{a_y} & r_{a_y} \\ n_{a_z} & b_{a_z} & t_{a_z} & r_{a_z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{cs} \\ y_{cs} \\ z_{cs} \\ 1 \end{bmatrix} \qquad (5.4)$$

Since $\underline{n}_a$, $\underline{b}_a$, $\underline{t}_a$ and $\underline{r}_a$ are vector functions of parameter θ_a and x_{cs} , y_{cs} and z_{cs} are functions of θ_a and θ_{cs} , the geometry of the cross section of the duct is governed by the value of θ_a.

5.3 Development of Duct

The portion of the duct between two cross sections can be approximated to a conical convolute constructed with the curves of these two cross sections acting as directrices. Depending upon the proximity of these two cross sections and the complexity of the geometry of the duct the accuracy of approximation will vary. If the variation in the shape and size of the cross section of the duct is not very abrupt and if the cross sections are close to each other, the approximation is better; otherwise it is poor. So depending upon the accuracy required and the complexity of the geometry of the duct, the duct is divided into a number of parts by planes passing through cross sections of interest along the duct. Each portion of the duct can be approximated to a conical convolute and then the entire duct can be considered to be a series of such convolutes put together as shown in Figure 5.2. The development of the duct is then achieved by developing these convolutes individually.

5.3.1 Series of Convolutes

If a duct is approximated by a series of n_d number of convolutes then the intermediate cross-sectional planes will be $(n_d - 1)$. Consequently, there will be $(n_d + 1)$ points along the centre line of the duct including the starting and end points. Let them be named as O_{kd}, $kd = 1, 2, 3, \ldots, (n_d + 1)$. The suffix kd corresponds to the serial number of the cross section under consideration, starting from the beginning of the duct.

Let

$$\underline{r}_{a_{kd}} \quad , \quad kd = 1,2,3,\ldots, (n_d + 1)$$

be the position vectors of these points O_{kd}. Then $\theta_{a_{kd}}$, the value of the parameter θ_a corresponding to these points O_{kd} can be found out from Eqn. (5.1) defining the centre line of the duct. If the complexity of the geometry of the duct permits equal division of the range of the parameter, then

$$\theta_{a_{kd}} = \theta_{a_s} + \frac{\theta_{a_f} - \theta_{a_s}}{n_d} (kd - 1) \quad . \tag{5.5}$$

In all cases,

$$\begin{aligned} &\theta_{a_1} = \theta_{a_s} \\ &\theta_{a_s} < \theta_{a_{kd}} < \theta_{a_f} \qquad 1 < kd < (n_d + 1) \\ &\theta_{a_{(n_d + 1)}} = \theta_{a_f} \end{aligned} \tag{5.6}$$

so that the entire duct from the starting point to the end point is approximated by a series of convolutes. Let $\underline{t}_{a_{kd}}$, $\underline{n}_{a_{kd}}$ and $\underline{b}_{a_{kd}}$ be the tangent, normal and bi-normal vectors corresponding to the point O_{kd}.

Also let the vector function

$$\underline{r}_{L_{kd}} = \begin{bmatrix} x_{cs}\,(\theta_{a_{kd}}, \theta_{cs}) \\ y_{cs}\,(\theta_{a_{kd}}, \theta_{cs}) \\ z_{cs}\,(\theta_{a_{kd}}, \theta_{cs}) \\ 1 \end{bmatrix} \tag{5.7}$$

define the kdth cross section of the duct (refer to Eqn. (5.3)) in local co-ordinates. Then in global co-ordinates, the kdth cross section is given by (refer to Eqn. (5.4))

$$\underline{r}_{kd} = \begin{bmatrix} n_{a_{kd_x}} & b_{a_{kd_x}} & t_{a_{kd_x}} & r_{a_{kd_x}} \\ n_{a_{kd_y}} & b_{a_{kd_y}} & t_{a_{kd_y}} & r_{a_{kd_y}} \\ n_{a_{kd_z}} & b_{a_{kd_z}} & t_{a_{kd_z}} & r_{a_{kd_z}} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{cs_{kd}} \\ y_{cs_{kd}} \\ z_{cs_{kd}} \\ 1 \end{bmatrix} \tag{5.8}$$

Considering two such curves of adjacent cross sections a conical convolute can be constructed. Each curve of the cross section except the first and the last one serves as primary as well as secondary directrix. What is secondary directrix to one convolute will be the primary directrix for the next convolute as

one proceeds along the duct from the starting point. The first curve acts as primary directrix to the first convolute only and the last curve acts as secondary directrix to the last convolute only. So for the jth convolute, jth and (j+1)th curves act as primary and secondary directrices respectively; j varies from 1 to n_d. Thus from the $(n_d + 1)$ curves of cross sections, n_d number of convolutes can be constructed. These form the series of convolutes which put together is the approximation for the geometry of the ideal duct surface.

5.3.2 Algorithm for the Development of Duct

It can be seen that the spatial configuration of all these conical convolutes are specified as per definition (B) given in Section 3.3.2. The development of these convolutes can be carried out as per the algorithm given in Section 3.4.8. The algorithm for the development of the duct is given below.

Step 1 : Read the values of θ_{a_s} , θ_{a_f} and n_d . Also read the details of the function defining the centre line of the duct.

Step 2 : Read the details of the function(s) defining the cross section of the duct.

Step 3 : Depending upon the complexity of the duct fix the points

O_{kd} , $kd = 1,2,3,\ldots, (n_d + 1)$

along the centre line of the duct. Find from Eqn. (5.1) the corresponding value of $\theta_{a_{kd}}$; or, if the range of parameter can be equally divided, then find the value of $\theta_{a_{kd}}$ from Eqn. (5.5).

Step 4 : Set $j = 1$. Corresponding to θ_{a_j} , calculate the vectors $\underline{r}_{a_j}$ (if not already known), $\underline{t}_{a_j}$, $\underline{n}_{a_j}$ and $\underline{b}_{a_j}$. Also calculate the vector $\underline{r}_{L_j}$. Now the details of the primary directrix for the jth convolute are available.

Step 5 : Similarly corresponding to $\theta_{a_{j+1}}$ calculate the vectors $\underline{r}_{a_{j+1}}$ (if not already known), $\underline{t}_{a_{j+1}}$, $\underline{n}_{a_{j+1}}$, $\underline{b}_{a_{j+1}}$ and $\underline{r}_{L_{j+1}}$. The details of the secondary directrix for the jth convolute are now available.

Step 6 : Carry out the development of the jth convolute as per the algorithm given in Section 3.4.8.

Step 7 : If $j = n_d$, stop. Otherwise set the details of the secondary directrix of the jth convolute as the details of the primary directrix of the $(j + 1)$th convolute. Set $j = j + 1$. Go to Step 5.

While following the above algorithm, the values of the parameter θ_{cs} for a curve of cross section may be

choosen in any one of the following two ways:

(i) The set of values of θ_{cs} used for a curve is the same whether it acts as primary curve for one convolute or as the secondary curve for the preceeding convolute. Thus the generic points along the curve are fixed as per the condition of developability when the curve acts as secondary directrix and the same points are used when the curve acts as primary directrix for the next convolute. Here the set of values of θ_{cs} differs for different curves,

(ii) The same set of values of θ_{cs} is used for all curves when they act as primary directrix. Thus the set of generic points considered along a curve when it acts as primary directrix are different from the set of generic points obtained as per condition of developability when the curve acts as secondary directrix.

Depending upon the situation, any one of the above methods can be used to fix the set of values of θ_{cs} for the various curves.

5.4 Case Study

Based on the algorithm given in the previous section, a computer programme has been developed. In

the programme, the details about the centre line of the duct and duct cross section are read through subroutine. Also the calculation of the position vector of a point on the centre line of the duct and the first and second derivatives at that point are calculated by a subroutine. The calculation of the details of the cross section at that point is done by yet another subroutine. Thus the programme is a general one and can be used for the development of a duct of a given centre line and cross section by suitably changing these subroutines and the relevant common, dimension and type declaration statements in the main programme. To illustrate the method two examples are presented here.

5.4.1 Example 5.1 : A Planar Duct

The duct considered here is a planar one (refer to figure 5.1). The centre line of the duct is a spiral and the cross section of the duct is circular in shape. The size of the cross section changes along the duct; from a given size at the starting point it reduces to zero at the end, in one complete rotation along the spiral. The duct is similar to the volute casing of hydraulic turbines and centrifugal pumps.

Let the centre line of the duct be in the O-XY plane. Let a_O be the radius of the circular cross section

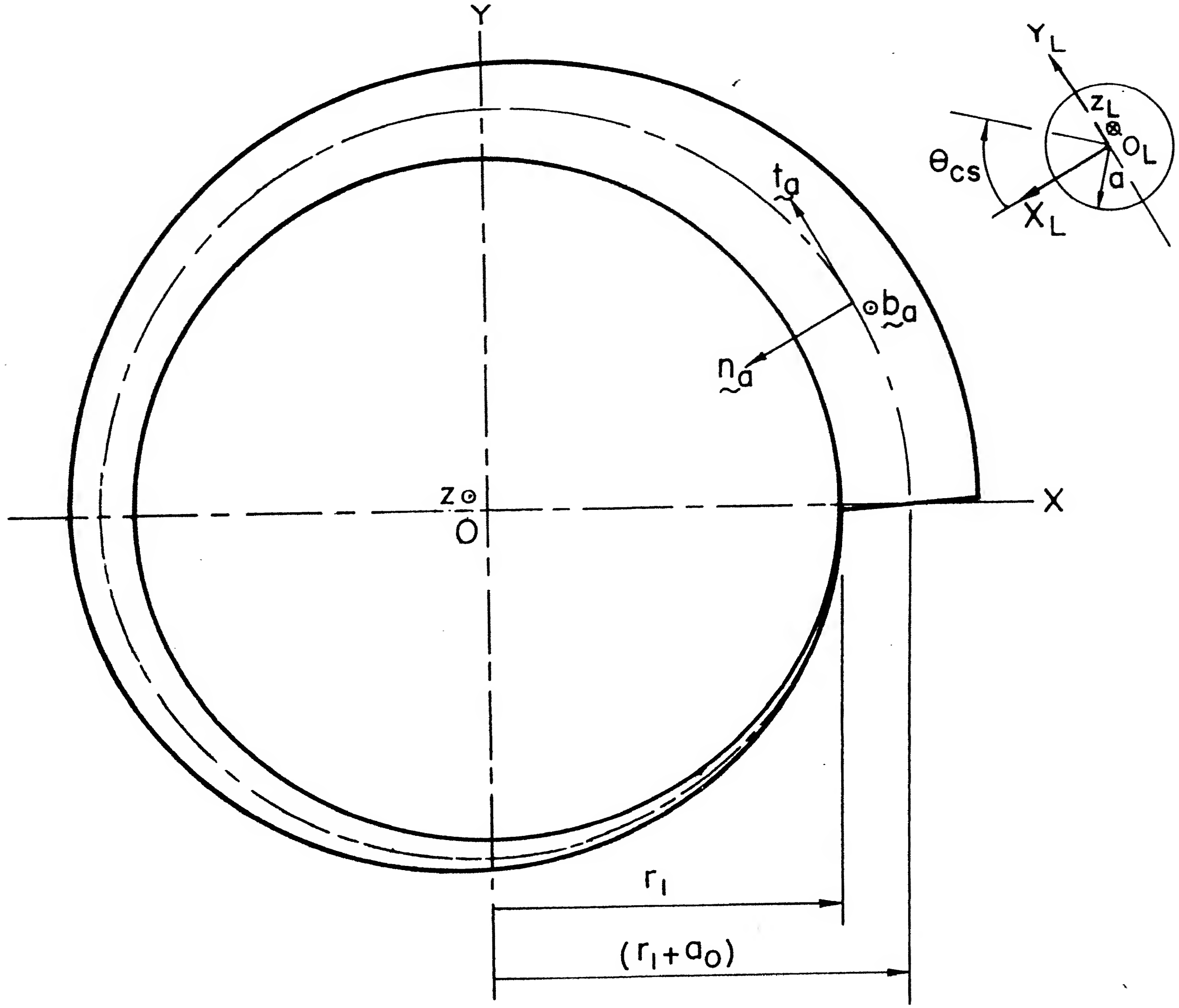


Fig. 5.1 Planar duct. Example 5.1

at the start. Then at any point along the duct, the radius of the cross section is given by

$$a = a_0 \, (1 - \theta_a/2\pi) \qquad 0 \leq \theta_a \leq 2\pi \tag{5.9}$$

where θ_a is the parameter of the centre line of the duct. Let the generic point on the centre line of the duct be given by

$$\underline{r}_a = \begin{bmatrix} r_2 \cos \theta_a \\ r_2 \sin \theta_a \\ 0 \\ 1 \end{bmatrix} \tag{5.10}$$

where

$$r_2 = r_1 + a_0 \, (1 - \theta_a/2\pi) \ . \tag{5.11}$$

Here r_2 is the distance of the generic point from the origin O ; at the starting point of the duct $r_2 = r_1 + a_0$ and at the end point $r_2 = r_1$. The tangent, normal and bi-normal vectors to the centre line are shown in the Figure 5.1. Also the cross section of the duct at the generic point is shown in the figure. A generic point of the curve of the cross section is given by

$$\underline{r}_{cs} = \begin{bmatrix} a \cos \theta_{cs} \\ a \sin \theta_{cs} \\ 0 \\ 1 \end{bmatrix} \quad 0 \leq \theta_{cs} \leq 2\pi \tag{5.12}$$

with respect to the local co-ordinate frame for the curve.

The following data is to be given as the input to the programme:

(i) the starting value of the parameter θ_a, θ_{a_s},

(ii) the final value of the parameter θ_a, θ_{a_f},

(iii) number of parts into which the duct is to be divided, n_d,

(iv) the value of r_1 and

(v) the value of a_0.

The actual data used in this example is

(i) $\theta_{a_s} = 0$

(ii) $\theta_{a_f} = 2\pi$

(iii) $n_d = 12$

(iv) $r_1 = 5.0$ and

(v) $a_0 = 1.0$

Of course always $\theta_{a_s} = 0$ and $\theta_{a_f} = 2\pi$. But if only a part of the duct (from the starting point corresponding to $\theta_{a_s} = 0$ to some point corresponding to a value of $\theta_{a_f} < 2\pi$) is considered, then the value of θ_{a_f} can be suitably given.

There are totally 13 cross sections, the details of which are given in Table 5.1. The input data for the various convolutes can be obtained from this table. The series of convolutes which is the approximation to the duct is shown in Figure 5.2. The serial number of the various cross sections and convolutes are given in the figure. The development of the various convolutes are given in Figure 5.3.

5.4.2 Example 5.2 : A Duct in Space

A duct in space is considered here. The centre line of the duct is a B'ezier curve defined by [12]

$$\underline{r}_a(\theta_a) = (1-\theta_a)^3\,\underline{r}_0 + 3\theta_a(1-\theta_a)^2\,\underline{r}_1 + 3\theta_a^2(1-\theta_a)\,\underline{r}_2 + \theta_a^3\,\underline{r}_3 \qquad (5.13)$$

where

$\underline{r}_0$ = position vector of starting point of the B'ezier curve,

$\underline{r}_3$ = position vector of the end point of the B'ezier curve,

$\underline{r}_1$ = position vector of a point on the tangent to the curve at the starting point and

$\underline{r}_2$ = position vector of a point on the tangent to the curve at the end point.

The range for θ_a is from 0 to 1.

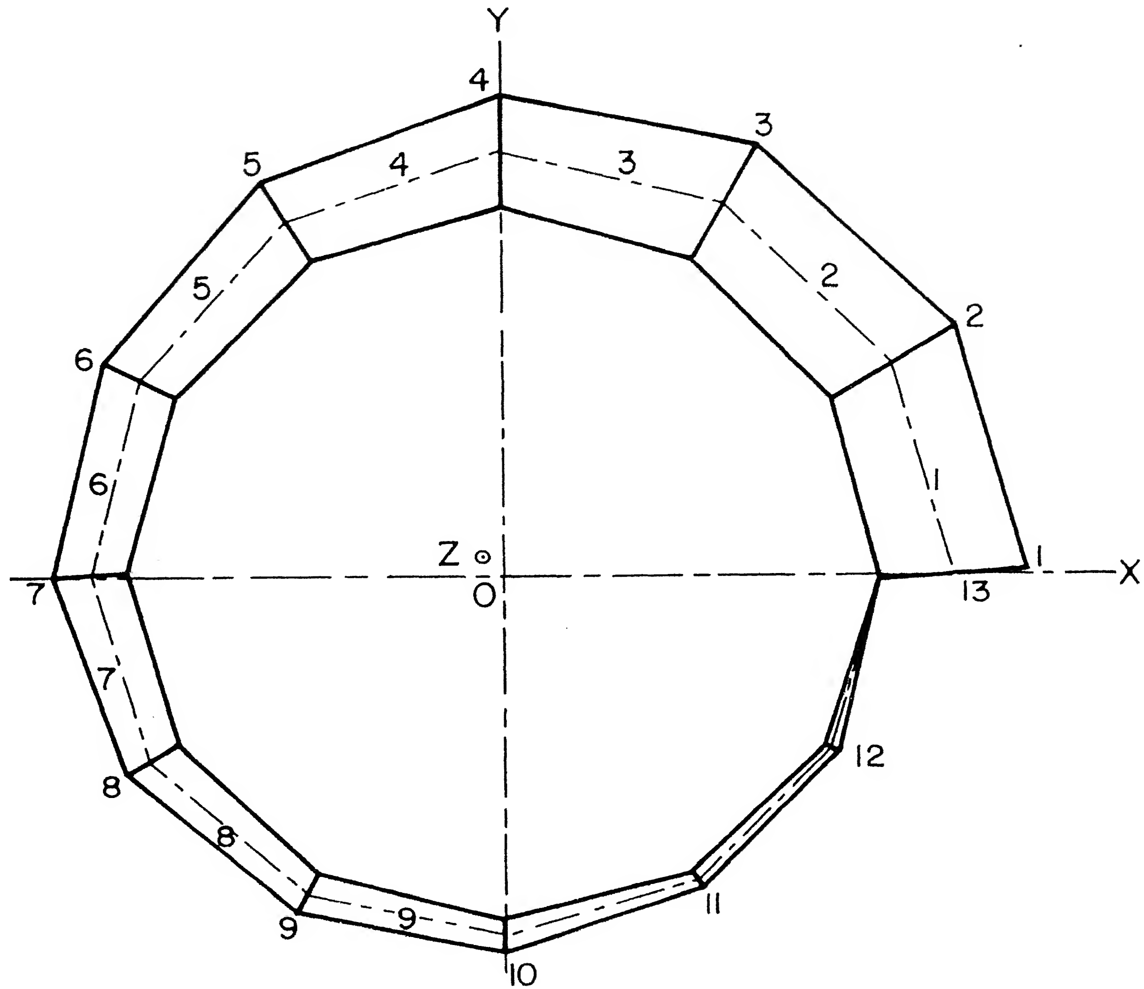
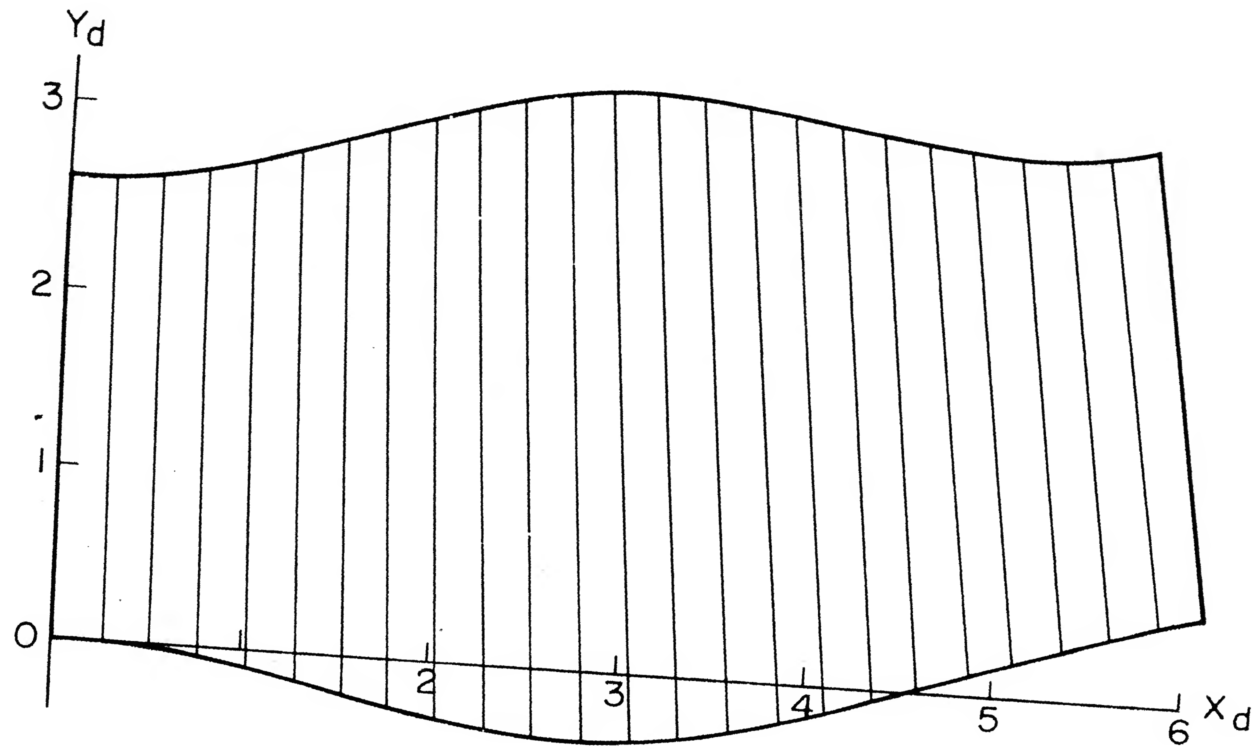
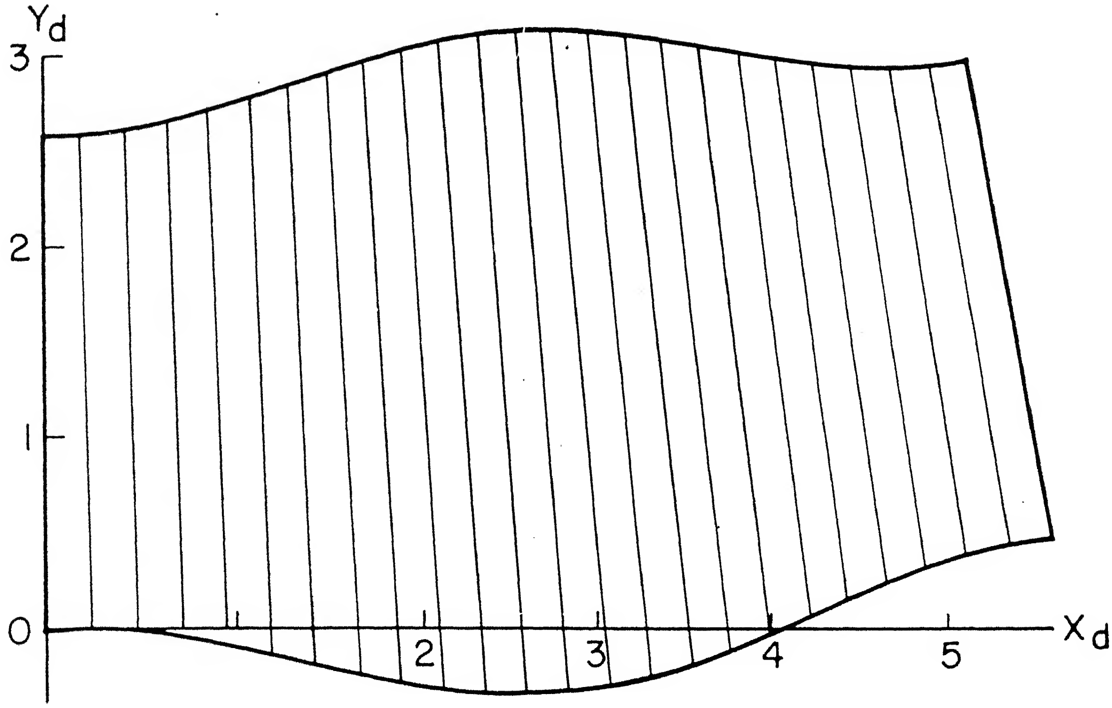


Fig.5.2 Series of conical convolutes. Approximation for the duct. Example 5.1.

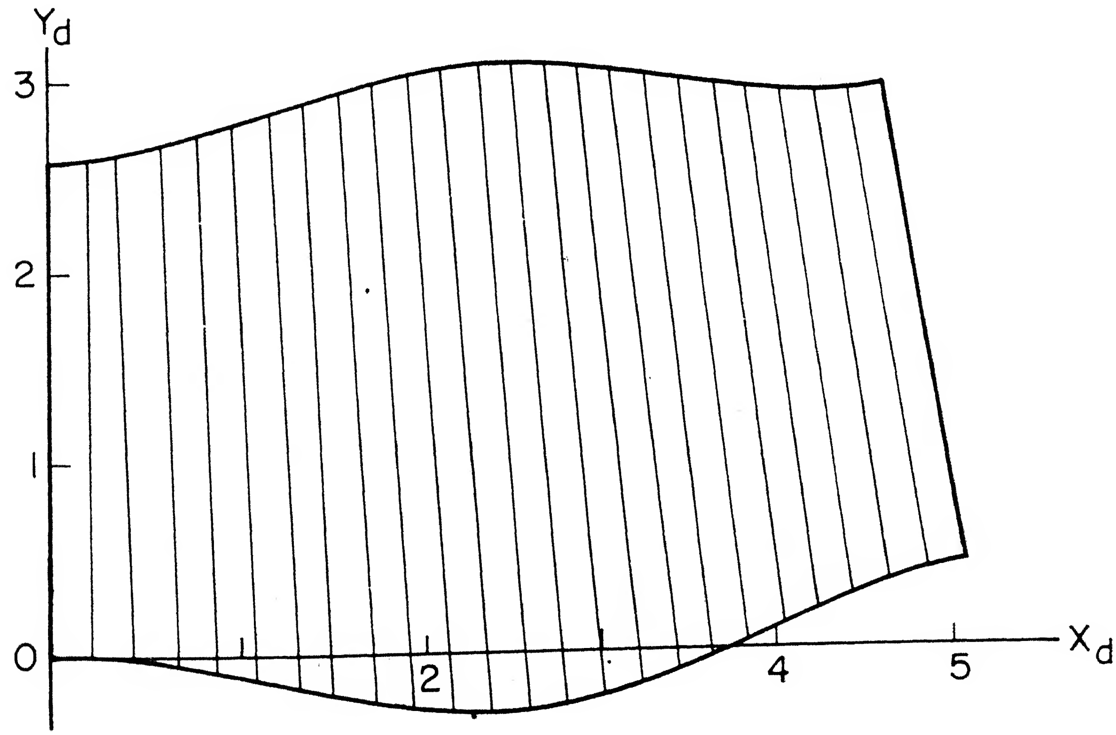


Convolute #1

Fig. 5.3 Development of the series of conical convolutes. Example 5.1

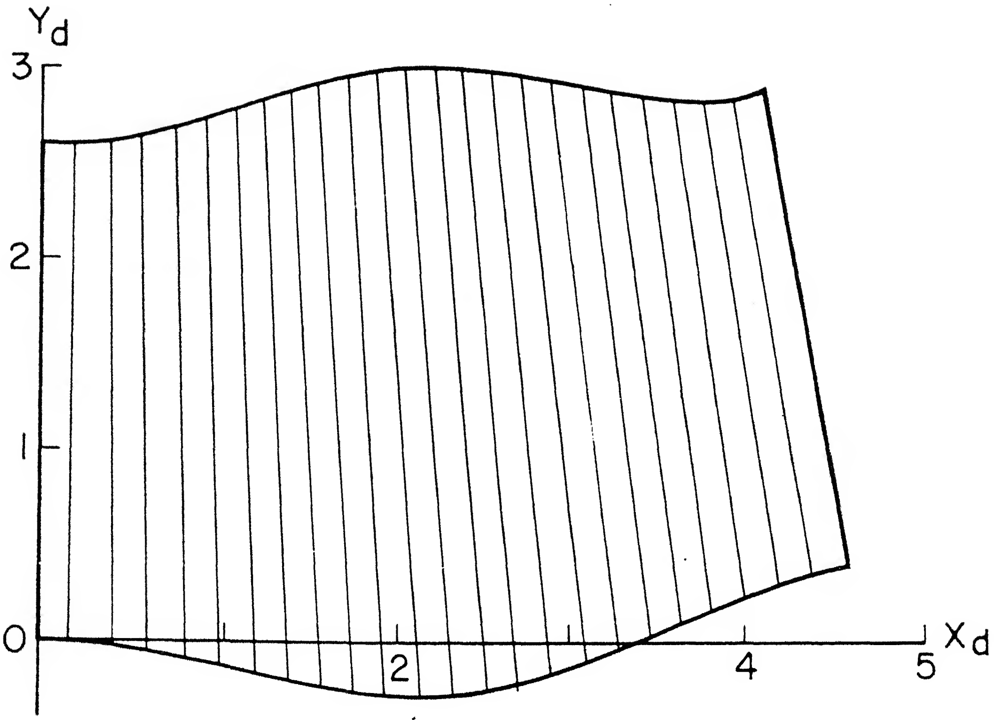


Convolute # 2

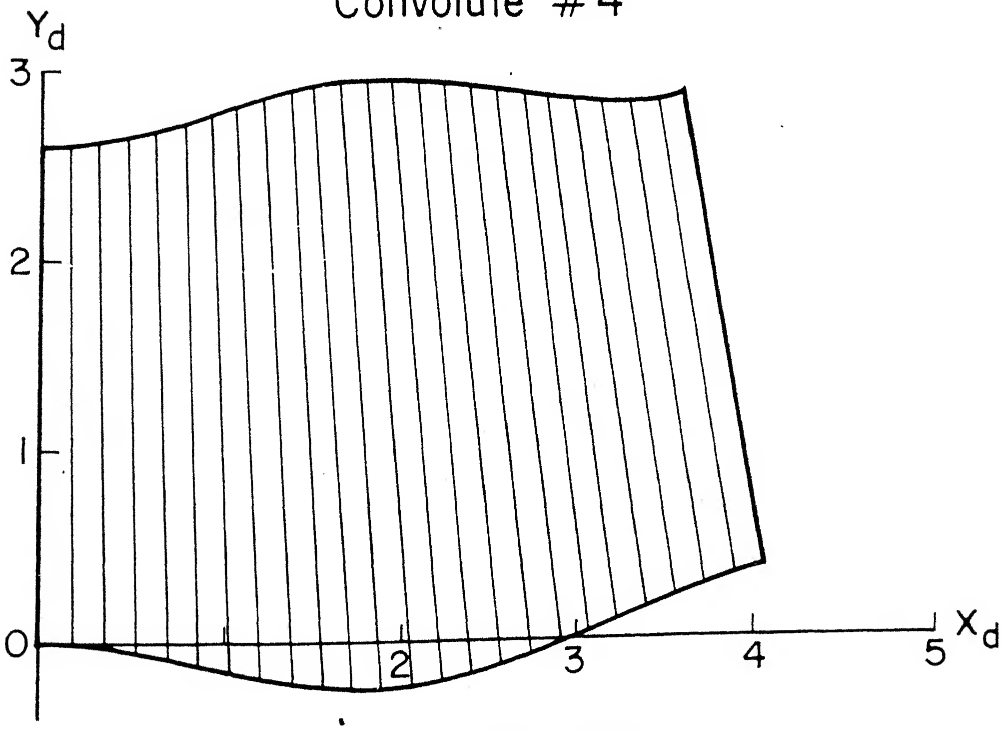


Convolute #3

Fig. 5.3 Development of the series of conical convolutes. Example 5.1

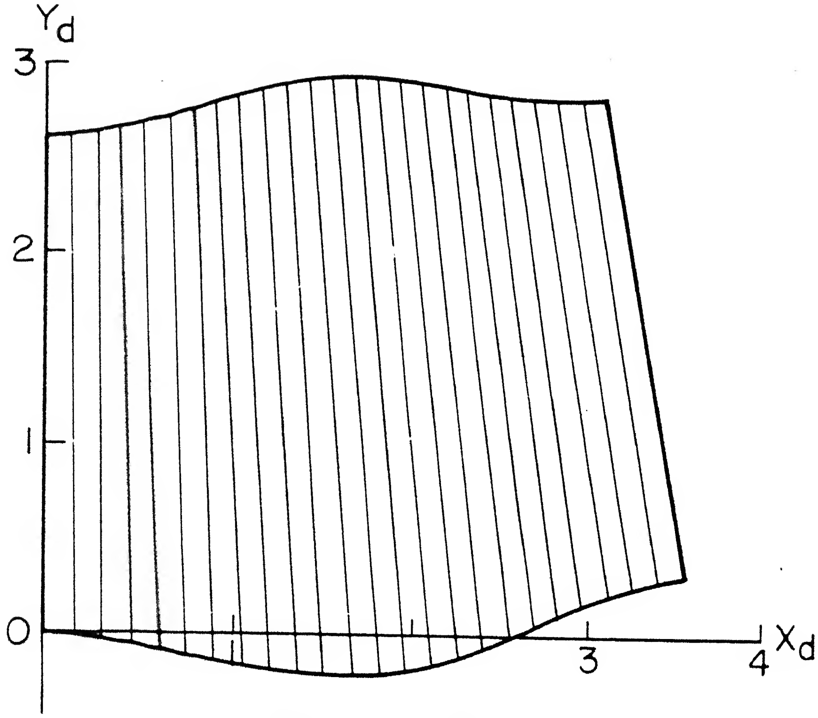


Convolute #4

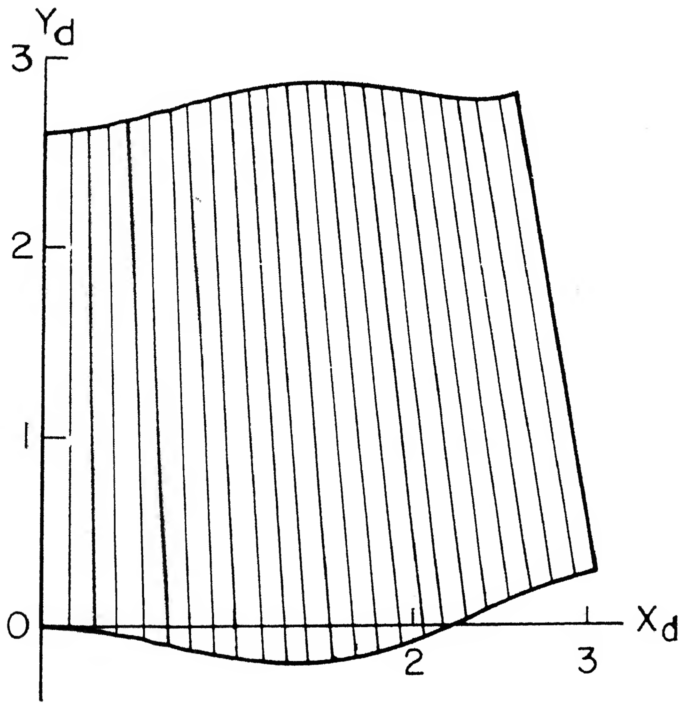


Convolute #5

Fig.5.3 Development of the series of conical convolutes. Example 5.1 .



Convolute # 6



Convolute # 7

Fig. 5.3 Development of the series of conical convolutes. Example 5.1.

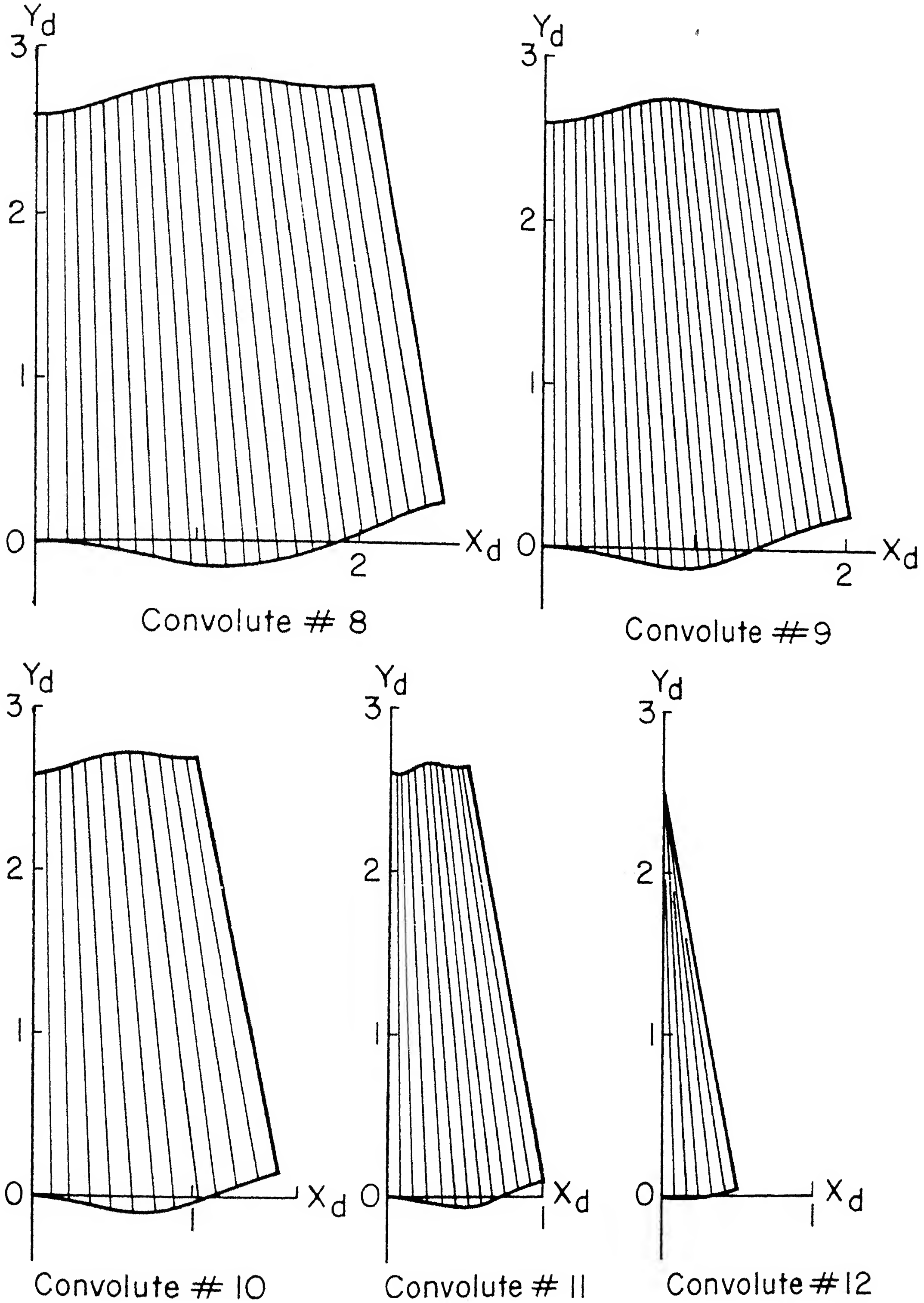


Fig. 5.3 Development of the series of conical convolutes. Example 5.1 .

The cross sections of the duct at the starting point and at the end point are taken to be a circle and a super-ellipse respectively. Since circle is a particular case of a super-ellipse, the cross section at any point, from the starting point to the end point, can be expressed as a super-ellipse defined by

$$\underline{r}_{cs} = \begin{bmatrix} a \cos^{2/n} \theta_{cs} \\ b \sin^{2/n} \theta_{cs} \\ 0 \\ 1 \end{bmatrix} \qquad (5.14)$$

in local co-ordinates of the cross section. Here the semi-major diameter a, the semi-minor diameter b and the power index n are to be controlled by the value of θ_a. They can be expressed as functions of θ_a and their values at the starting and end points of the duct. Thus

$$\begin{aligned} a &= f_1 (a_0 , a_f , \theta_a) \\ b &= f_2 (b_0 , b_f , \theta_a) \\ n &= f_3 (n_0 , n_f , \theta_a) \end{aligned} \qquad (5.15)$$

where f_1 , f_2 and f_3 are functions which can be formulated to suit the nature of variation in shape and size of the cross section along the duct. The subscripts 0 and f are used to indicate the starting and end points respectively.

The input details required are:

(i) the position vectors of the four points P_0, P_1, P_2 and P_3 defining the Bézier curve, namely $\underline{r}_0$, $\underline{r}_1$, $\underline{r}_2$ and $\underline{r}_3$,

(ii) the values of a_0, b_0, n_0, a_f, b_f and n_f,

(iii) the three functions f_1, f_2 and f_3 and

(iv) the manner and the number of parts, n_d, into which the duct is to be divided for the purpose of development.

Once the three functions f_1, f_2 and f_3 are defined, the relevant subroutine to find the details of the cross section of the duct corresponding to a given value of the parameter θ_a can be written.

Following input data is used in the example presented here:

$$\underline{r}_0 = \begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 1 \end{bmatrix} ; \underline{r}_1 = \begin{bmatrix} 2.5 \\ 0.0 \\ 0.0 \\ 1 \end{bmatrix} ; \underline{r}_2 = \begin{bmatrix} 5.0 \\ 2.5 \\ 5.0 \\ 1 \end{bmatrix} ; \underline{r}_3 = \begin{bmatrix} 5.0 \\ 5.0 \\ 5.0 \\ 1 \end{bmatrix}$$

$a_0 = 0.5$; $b_0 = 0.5$; $n_0 = 2$

$a_f = 1.0$; $b_f = 0.6$; $n_f = 4$

$n_d = 5$; the range of parameter θ_a is to be divided equally.

The functions f_1 , f_2 and f_3 are choosen to be

$$\begin{aligned} f_1 &: a = a_0 + (a_f - a_0)\,\theta_a \;, \\ f_2 &: b = b_0 + (b_f - b_0)\,\theta_a \;, \\ f_3 &: n = n_0 + (n_f - n_0)\,\theta_a \;. \end{aligned} \tag{5.16}$$

This means linear variation of a, b and n with respect to the parameter θ_a.

The values of the parameter θ_a considered here are 0.,0.2, 0.4, 0.6, 0.8 and 1.0. The details of the corresponding cross sections of the duct are given in Table 5.2. The centre line of the duct is shown in Figure 5.4. The points P_0 , P_1 , P_2 , P_3 defining the Bézier curve and the points along the centre line corresponding to the choosen values of θ_a are also shown. The cross section of the duct at various points along its centre line are shown in Figure 5.5. The development of the super-conical convolutes which are the approximations of the parts of the duct are shown in Figure 5.6.

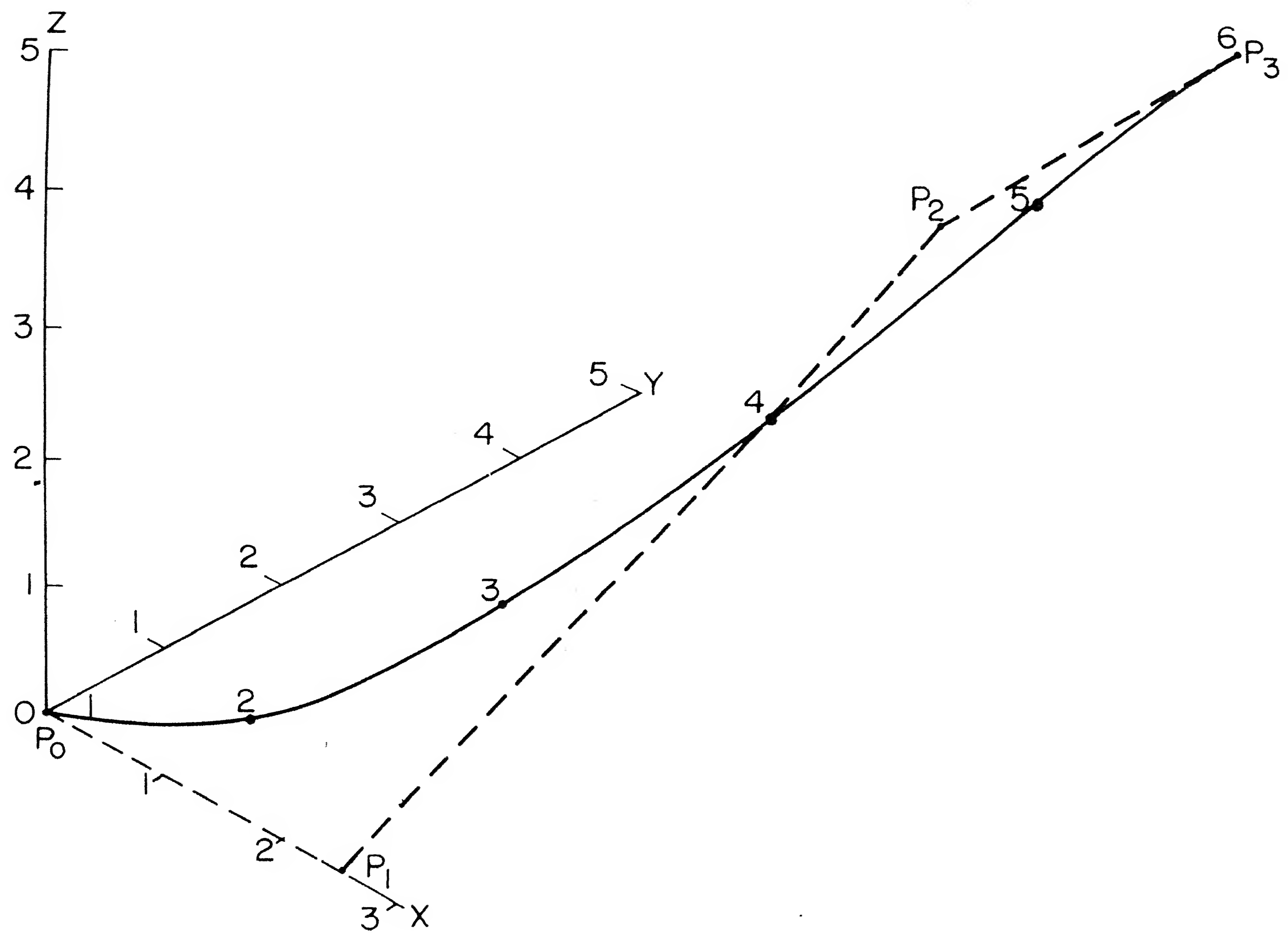


Fig. 5.4 Centre line of duct in space. Example 5.2.

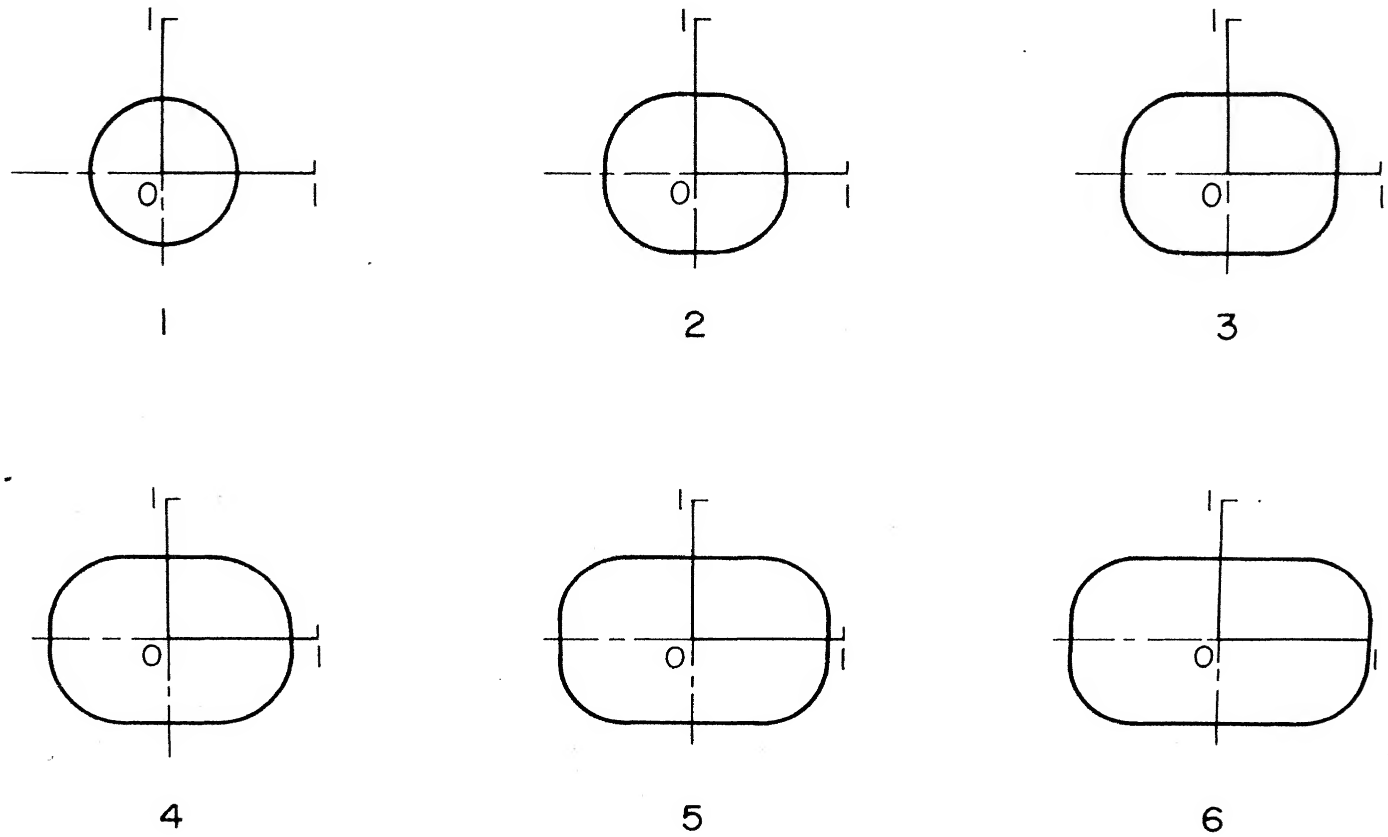


Fig.5.5 Cross sections of the duct in space. Example 5.2.

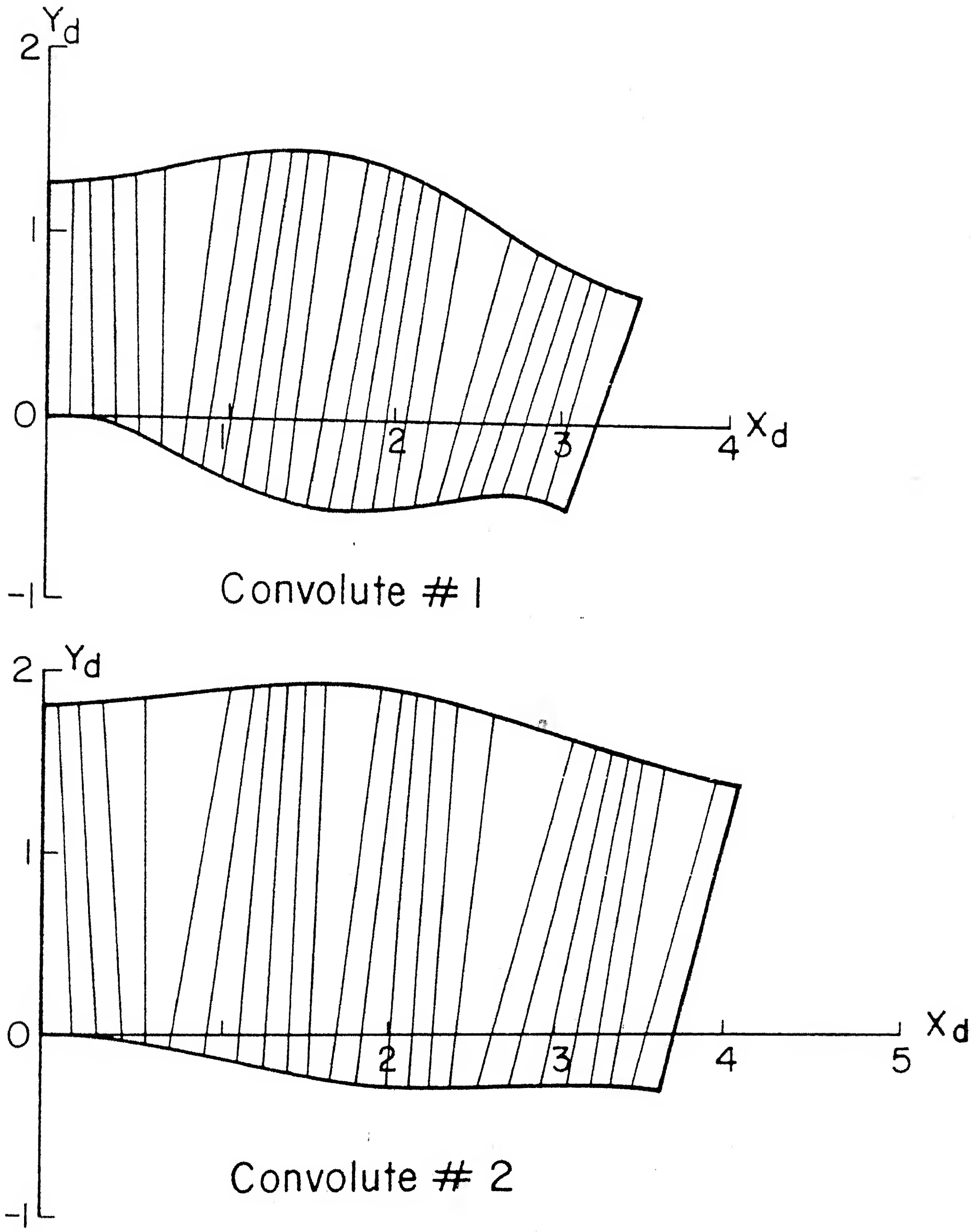
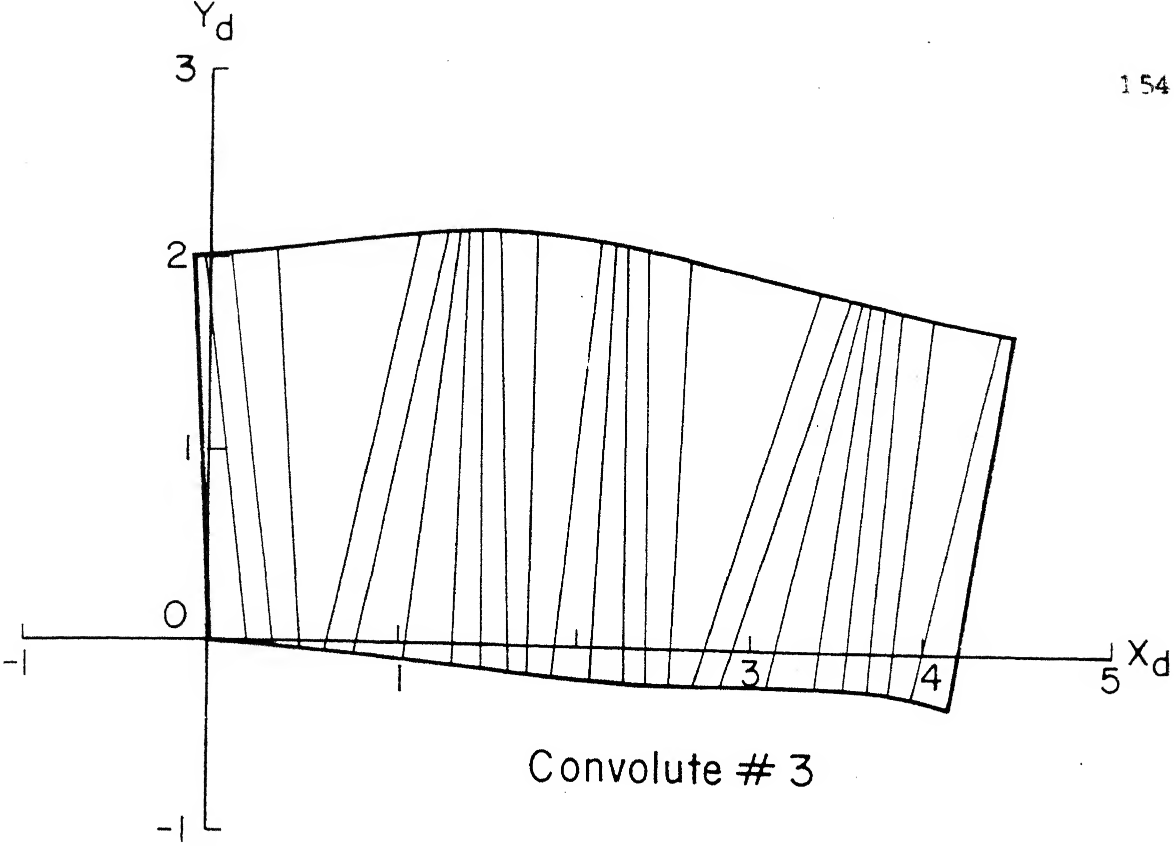
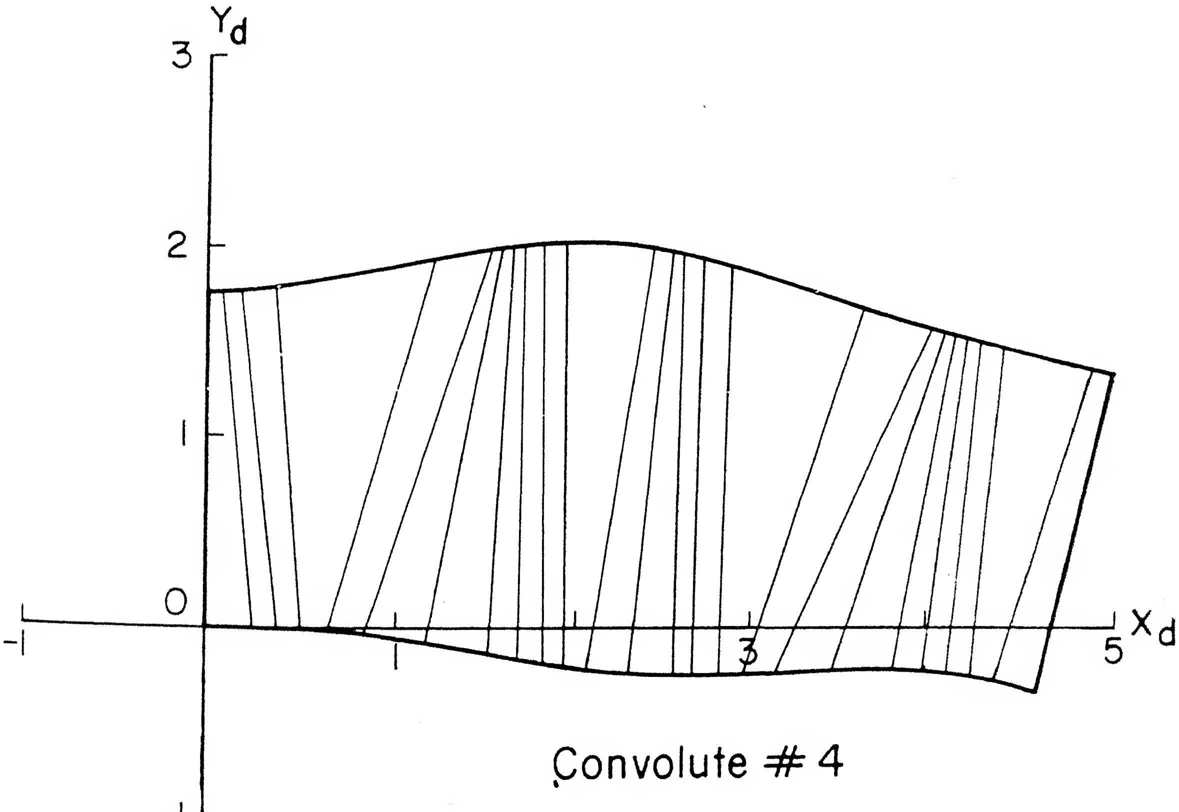


Fig. 5.6 Development of the duct in space.
Example 5.2

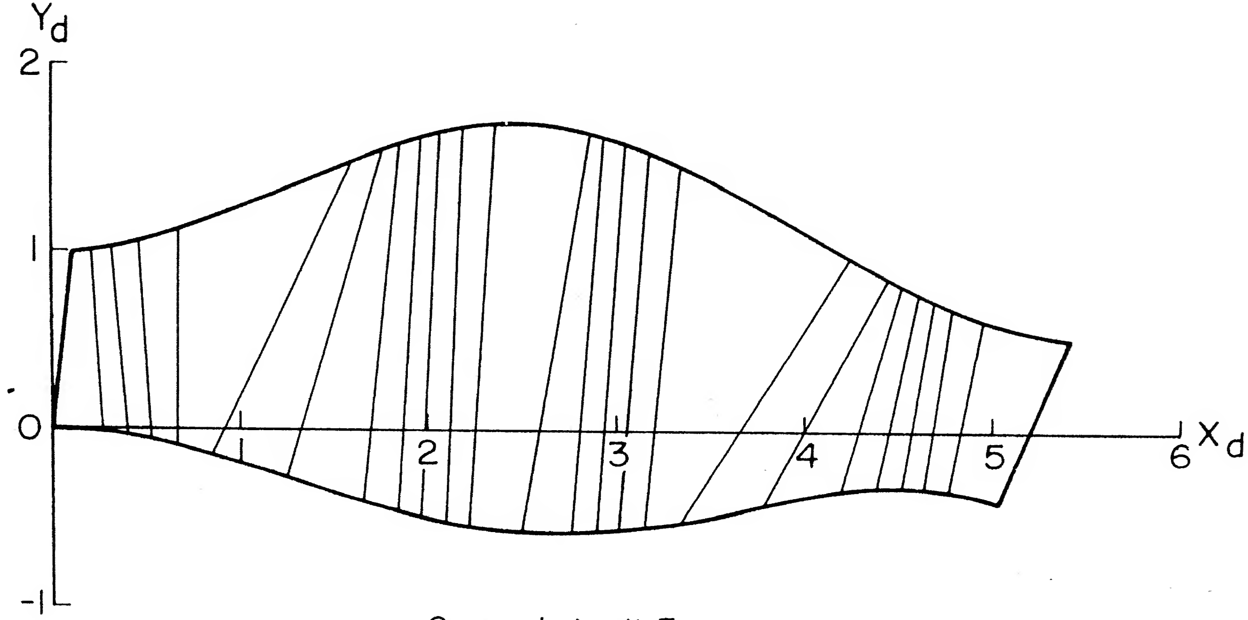


Convolute # 3



Convolute # 4

Fig. 5.6 Development of the duct in space. Example 5.2.



Convolute # 5

Fig.5.6 Development of the duct in space. Example 5.2.

Table 5.1 Details of Cross Sections of Planar Duct. Example 5.1

S.No.	θ_a, deg	$\underline{r}_a$	$\underline{t}_a$	$\underline{n}_a$	$\underline{b}_a$	a
1	0.	$\begin{bmatrix} 6.0 \\ 0.0 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} -0.0265 \\ 0.9996 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} -0.9996 \\ -0.0265 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix}$	1.0
2	30.	$\begin{bmatrix} 5.124 \\ 2.958 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} -0.5231 \\ 0.8523 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} -0.8523 \\ -0.5231 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix}$	0.9167
3	60.	$\begin{bmatrix} 2.917 \\ 5.052 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} -0.8793 \\ 0.4762 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} -0.4762 \\ -0.8793 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix}$	0.8333
4	90.	$\begin{bmatrix} 0.0 \\ 5.75 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} -0.9996 \\ -0.0277 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.0277 \\ -0.9996 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix}$	0.75
5	120.	$\begin{bmatrix} -2.833 \\ 4.907 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} -0.8516 \\ -0.5241 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.5241 \\ -0.8516 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix}$	0.6667
6	150.	$\begin{bmatrix} -4.835 \\ 2.792 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} -0.4751 \\ -0.8799 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.8799 \\ -0.4751 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix}$	0.5833
7	180.	$\begin{bmatrix} -5.5 \\ 0.0 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.0289 \\ -0.9996 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.9996 \\ 0.0289 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix}$	0.5
8	210	$\begin{bmatrix} -4.691 \\ -2.708 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.5252 \\ -0.851 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.851 \\ 0.5252 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix}$	0.4167

Contd.....

Contd.... Table 5.1 Details of Cross Sections of Planar Duct. Example 5.1

S.No.	θ_a, deg	$\underline{r}_a$	$\underline{t}_a$	$\underline{n}_a$	$\underline{b}_a$	a
9	240.	$\begin{bmatrix} -2.667 \\ -4.619 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.8806 \\ -0.4739 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.4739 \\ 0.8806 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix}$	0.333
10	270.	$\begin{bmatrix} 0.0 \\ -5.25 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.9995 \\ 0.0303 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} -0.0303 \\ 0.9995 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix}$	0.25
11	300.	$\begin{bmatrix} 2.583 \\ -4.474 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.8502 \\ 0.5264 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} -0.5264 \\ 0.8502 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix}$	0.167
12	330.	$\begin{bmatrix} 4.402 \\ -2.542 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.4727 \\ 0.8812 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} -0.8812 \\ 0.4727 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix}$	0.083
13	360.	$\begin{bmatrix} 5.0 \\ 0.0 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} -0.0318 \\ 0.9995 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} -0.9995 \\ -0.0318 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix}$	0.0

Table 5.2 Details of Cross Sections of Duct in Space. Example 5.2

S.No.	θ_a	$\underline{r}_a$	$\underline{t}_a$	$\underline{n}_a$	$\underline{b}_a$	a	b	n
1	0.0	$\begin{bmatrix}0.0\\0.0\\0.0\end{bmatrix}$	$\begin{bmatrix}1.0\\0.0\\0.0\end{bmatrix}$	$\begin{bmatrix}0.0\\0.447\\0.894\end{bmatrix}$	$\begin{bmatrix}0.0\\-0.394\\0.447\end{bmatrix}$	0.5	0.5	2.0
2	0.2	$\begin{bmatrix}1.48\\0.28\\0.52\end{bmatrix}$	$\begin{bmatrix}0.794\\0.298\\0.530\end{bmatrix}$	$\begin{bmatrix}-0.605\\0.463\\0.648\end{bmatrix}$	$\begin{bmatrix}-0.052\\-0.835\\0.548\end{bmatrix}$	0.6	0.52	2.4
3	0.4	$\begin{bmatrix}2.84\\1.04\\1.76\end{bmatrix}$	$\begin{bmatrix}0.589\\0.448\\0.673\end{bmatrix}$	$\begin{bmatrix}-0.754\\0.605\\0.256\end{bmatrix}$	$\begin{bmatrix}-0.292\\-0.658\\0.694\end{bmatrix}$	0.7	0.54	2.8
4	0.6	$\begin{bmatrix}3.96\\2.16\\3.24\end{bmatrix}$	$\begin{bmatrix}0.448\\0.589\\0.673\end{bmatrix}$	$\begin{bmatrix}-0.605\\0.754\\-0.256\end{bmatrix}$	$\begin{bmatrix}-0.658\\-0.292\\0.694\end{bmatrix}$	0.8	0.56	3.2
5	0.8	$\begin{bmatrix}4.72\\3.52\\4.48\end{bmatrix}$	$\begin{bmatrix}0.298\\0.794\\0.530\end{bmatrix}$	$\begin{bmatrix}-0.463\\0.605\\-0.648\end{bmatrix}$	$\begin{bmatrix}-0.835\\-0.052\\0.548\end{bmatrix}$	0.9	0.58	3.6
6	1.0	$\begin{bmatrix}5.0\\5.0\\5.0\end{bmatrix}$	$\begin{bmatrix}0.0\\1.0\\0.0\end{bmatrix}$	$\begin{bmatrix}-0.447\\0.0\\-0.894\end{bmatrix}$	$\begin{bmatrix}-0.894\\0.0\\0.447\end{bmatrix}$	1.0	0.6	4.0

Chapter 6

DEVELOPMENT OF THICK SURFACES

6.1 Introduction

In this chapter the development of uniformly thick surfaces is considered. A thick surface can be considered to be a set of layers of thin surfaces. Out of these thin surfaces, a mean surface can be considered such that at any point on it the top and bottom layers of the thick surface are at equal distance. The mean surface thus divides the thick surface into two halves of equal thicknesses. Now each of these two slices can further be divided into a number of slices of uniform thickness. The set of surfaces that separate these slices from one another form the thick surface. Ofcourse the top and bottom surfaces of the thick surface are also included in this set of surfaces. These surfaces are parallel to one another.

Here the mean surface should be a developable ruled surface. All other surfaces in the set are also shown to be developable ruled surfaces. These surfaces are developed individually and then these developments

are stacked together so that the development of the thick surface is obtained.

6.2 The Set of Surfaces

The set of surfaces which form the thick surface is shown in Figure 6.1. Let E_1, E_2, E_3 and E_4 be the end surfaces of the thick surface. These end surfaces bound the set of surfaces. The edges of the various surfaces in the set are the lines of intersection of these surfaces with the end surfaces. For example the edges of the mean surface are the lines of intersection of the mean surface with the end surfaces.

Let h be thickness of the thick surface and

s_h be number of slices into which each half of the thick surface is to be divided.

Then the total number of slices is $2s_h$ and the total number of surfaces in the set of surfaces representing the thick surface is $2s_h + 1$. Let kt = 1, 2, 3,...., $(2s_h + 1)$ be the serial number for the surfaces. Let kt = 1 indicate the mean surface. The direction of the inward normal or that of the outward normal to the mean surface can be taken as positive. Then let kt = 2, 3,...., $(s_h + 1)$ and kt = $(s_h + 2)$, $(s_h + 3)$,....,$(2s_h + 1)$ represent the surfaces in the positive and negative direction respectively of the normal in the increasing order of distance from the mean surface.

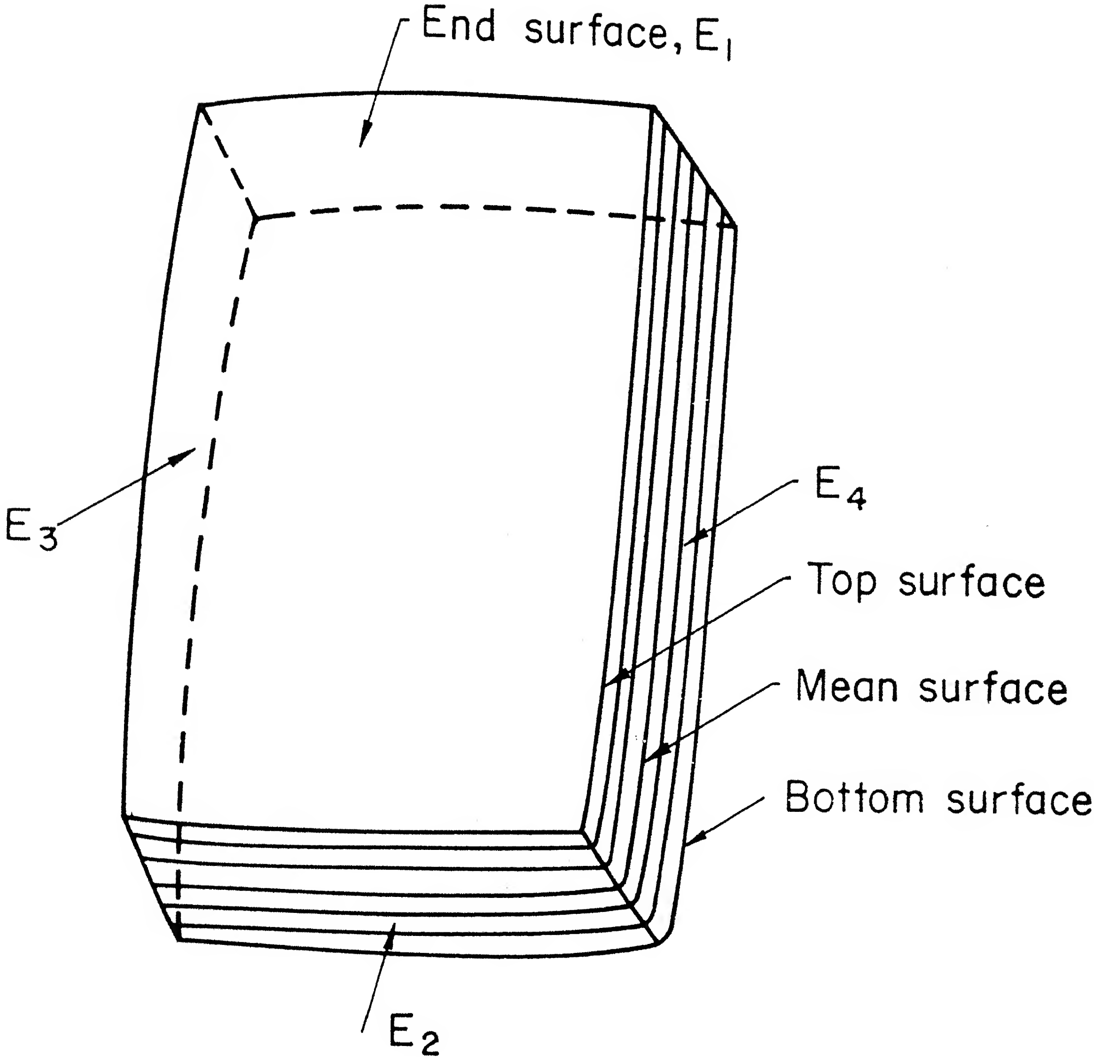


Fig.6.1 Set of surfaces.

Let c_{kt} , $kt = 1, 2, \ldots, (2s_h + 1)$ be the distance of the various surfaces S_{kt} from the mean surface. Ofcourse $c_1 = 0$ since $kt = 1$ represents the mean surface. The distance is positive if it is along the positive direction of the normal to the mean surface. Otherwise it is negative. If the slices are of equal thickness, then

$$
\begin{aligned}
c_1 &= 0 \\
c_{kt} &= \frac{h}{2s_h}(kt - 1) \qquad kt = 2, 3, \ldots, s_h + 1 \\
c_{kt} &= \frac{-h}{2s_h}(kt - s_h - 1) \qquad kt = (s_h + 2), (s_h + 3), \ldots, (2s_h + 1)
\end{aligned}
\tag{6.1}
$$

6.2.1 The Mean Surface

Consider the mean surface S_1. Since the mean surface is considered to be a ruled surface, any two of its edges can be considered to be its primary and secondary directrices. Let, for example, the lines of intersection of the mean surface with the end surfaces E_1 and E_2 be respectively the primary and secondary directrices. Let $\underline{r}_{i,1}$ and $\underline{r}_{j,1}$ be the position vector of the generic points on the primary and secondary directrices respectively. Here the suffix 1 stands for the mean surface and the suffices i and j stand for the primary

and secondary directrices respectively. If θ_i and θ_j are the parameters for the primary and secondary directrices respectively, then the condition for the developability of the mean surface is given by

$$\left(\frac{d\underline{r}_{i,1}}{d\theta_i} \times \frac{d\underline{r}_{j,1}}{d\theta_j}\right) \cdot (\underline{r}_{j,1} - \underline{r}_{i,1}) = 0 \quad . \qquad (6.2)$$

By solving the above equation for various values of θ_i, the generatrices of the mean surface can be fixed. One such generatrix, $P_1 Q_1$, is shown in Figure 6.2. The tangents $\frac{d\underline{r}_{i,1}}{d\theta_i}$ and $\frac{d\underline{r}_{j,1}}{d\theta_j}$, the unit normal to the mean surface, $\underline{n}_s$ and the unit vector $\underline{g}$ along the generatrix $P_1 Q_1$ are all shown in the figure. The unit vector $\underline{g}$ is the directional vector of the generatrix and is given by

$$\underline{g} = \frac{\underline{r}_{j,1} - \underline{r}_{i,1}}{[(\underline{r}_{j,1} - \underline{r}_{i,1}) \cdot (\underline{r}_{j,1} - \underline{r}_{i,1})]^{1/2}} \quad . \qquad (6.3)$$

Also shown in the figure is a unit vector $\underline{e}$ which together with the vectors $\underline{g}$ and $\underline{n}_s$ form a right-handed system of mutually perpendicular unit vectors.

$$\underline{e} = \underline{g} \times \underline{n}_s \quad . \qquad (6.4)$$

With the point P_1 as origin and the direction of the vectors $\underline{g}$, $\underline{n}_s$ and $\underline{e}$ as the X, Y and Z axes a local co-ordinate frame $P_1 - \underline{g}\underline{n}_s\underline{e}$ can be considered. Similarly another local co-ordinate frame $Q_1 - \underline{g}\underline{n}_s\underline{e}$ can be considered.

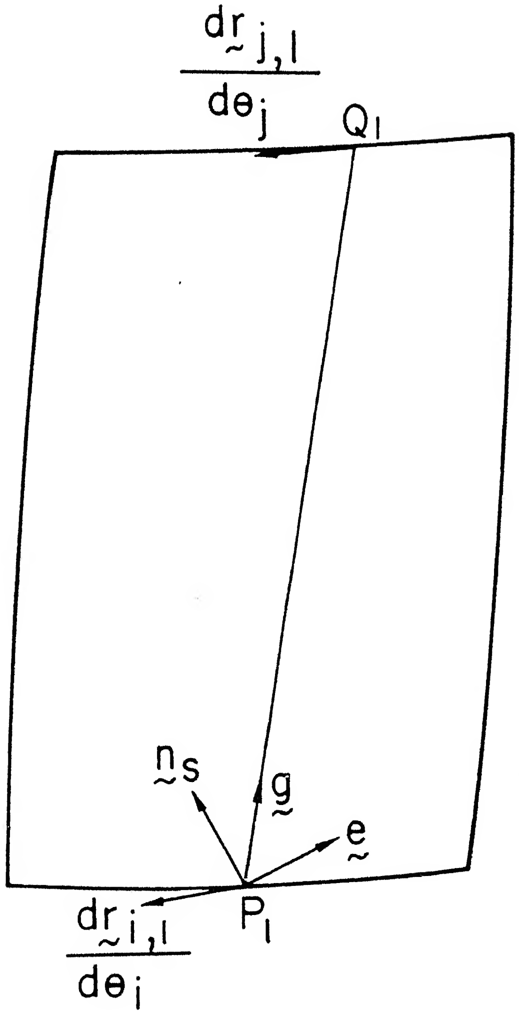


Fig.6.2 Mean surface.

6.2.2 Derivatives of Vectors $\underline{g}$ and $\underline{n}_s$

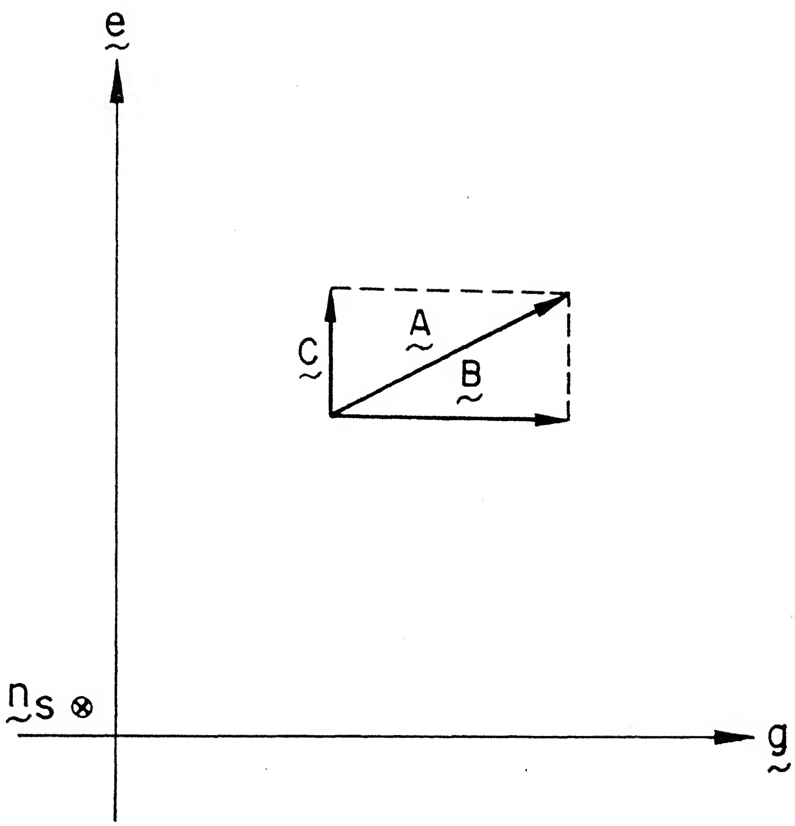
The unit vectors $\underline{g}$ and $\underline{n}_s$ can be considered to have either θ_i or θ_j as their parameter since θ_i and θ_j are interrelated through the condition for developability of the mean surface (Eqn. (6.2)). Differentiating the vector $\underline{g}$ with respect to θ_i and simplifying,

$$\frac{d\underline{g}}{d\theta_i} = \frac{\left(\frac{d\theta_j}{d\theta_i}\frac{d\underline{r}_{j,1}}{d\theta_j} - \frac{d\underline{r}_{i,1}}{d\theta_i}\right) - \left\{\left(\frac{d\theta_j}{d\theta_i}\frac{d\underline{r}_{j,1}}{d\theta_j} - \frac{d\underline{r}_{i,1}}{d\theta_i}\right).\,\underline{g}\right\}\underline{g}}{[(r_{j,1} - r_{i,1}).(r_{j,1} - r_{i,1})]^{1/2}} \quad \ldots\ldots (6.5)$$

From the above equation it can be seen that the dot product of $d\underline{g}/d\theta_i$ with $\underline{g}$ is zero and hence $d\underline{g}/d\theta_i$ and $\underline{g}$ are mutually perpendicular. Also vectors $d\underline{r}_{j,1}/d\theta_j$, $d\underline{r}_{i,1}/d\theta_i$ and $\underline{g}$ are all vectors in the $\underline{g}$-$\underline{e}$ plane and are perpendicular to $\underline{n}_s$. Hence $d\underline{g}/d\theta_i$ is perpendicular to $\underline{n}_s$. Thus the vector $d\underline{g}/d\theta_i$ is perpendicular to both the vectors $\underline{g}$ and $\underline{n}_s$. Hence it is along the vector $\pm\,\underline{e}$ depending upon the direction of the vector expressed by the numerator of Eqn. (6.5) (refer to Figure 6.3).

Similarly the derivative of $\underline{g}$ with respect to the parameter θ_j is given by

$$\frac{d\underline{g}}{d\theta_j} = \frac{\left(\frac{d\underline{r}_{j,1}}{d\theta_j} - \frac{d\theta_i}{d\theta_j}\frac{d\underline{r}_{i,1}}{d\theta_i}\right) - \left\{\left(\frac{d\underline{r}_{j,1}}{d\theta_j} - \frac{d\theta_i}{d\theta_j}\frac{d\underline{r}_{i,1}}{d\theta_i}\right).\underline{g}\right\}\underline{g}}{[(\underline{r}_{j,1} - \underline{r}_{i,1})\cdot(\underline{r}_{j,1} - \underline{r}_{i,1})]^{1/2}} \quad \ldots\ldots (6.6)$$



$$\underset{\sim}{A} = \frac{d\theta_j}{d\theta_i}\,\frac{d\underset{\sim}{r}_{j,l}}{d\theta_j} - \frac{d\underset{\sim}{r}_{i,l}}{d\theta_i}$$

$$\underset{\sim}{B} = (\underset{\sim}{A} \cdot \underset{\sim}{g})\,\underset{\sim}{g}$$

$$\underset{\sim}{C} = \underset{\sim}{A} - \underset{\sim}{B}$$

Fig. 6.3 Derivative of $\underset{\sim}{g}$

and it is along $\pm$ e depending upon the sign of the vector expressed by the numerator (of Eqn. (6.6)).

Since the vector $\underline{n}_s$ is a unit vector, its derivative with respect to the parameter θ_i or θ_j is perpendicular to it. Hence the vectors $d\underline{n}_s/d\theta_i$ and $d\underline{n}_s/d\theta_j$ are in the $\underline{g}$-$\underline{e}$ plane.

6.2.3 Other Surfaces

Other surfaces in the set of surfaces are defined to be parallel to the mean surface at any point. Corresponding to the generatrix $P_1\ Q_1$ of the mean surface there will be a straight line parallel to $P_1\ Q_1$ on each of these surfaces. These are the lines of intersection of these surfaces with the $\underline{g}$-$\underline{n}_s$ plane and the distance between them and the line $P_1\ Q_1$ is the distance between the corresponding surface and the mean surface.

Consider one of these surfaces S_{kt} and let $P_{kt}\,Q_{kt}$ be the straight line on this surface which is parallel to the generatrix $P_1\ Q_1$ of the mean surface (refer to Figure 6.4). Let P_{kt} and Q_{kt} be the points of intersection of the line $P_{kt}\ Q_{kt}$ with the end surfaces E_1 and E_2 respectively. The distance between this surface and the mean surface is c_{kt}. Let $O_{L1} - X_{L1}\ Y_{L1}\ Z_{L1}$ be a local co-ordinate frame such that $X_{L1}\ Y_{L1}$ is the plane of the end surface E_1 and Z_{L1} axis is normal to it.

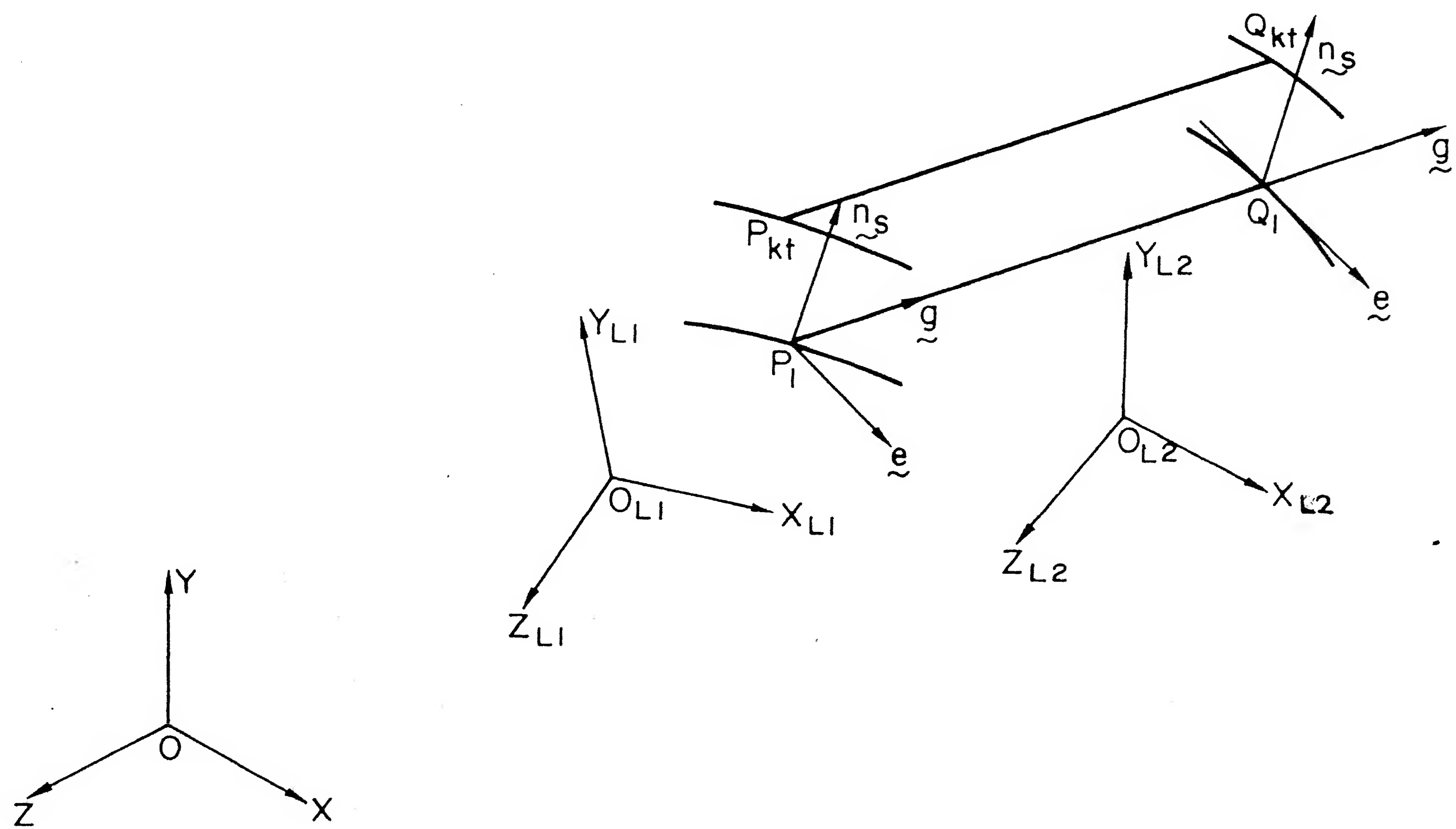


Fig. 6.4 Other surfaces.

Let O-XYZ be the global co-ordinate frame.

Let the position vector of the point P_{kt} be $[x, y, z, 1]$, $[x_1, y_1, 0, 1]$ and $[x^*, c_{kt}, 0, 1]$ with respect to the three co-ordinate frames O-XYZ, $O_{L1} - X_{L1} Y_{L1} Z_{L1}$ and $P_1 - \underline{g}\ \underline{n}_s\ \underline{e}$ respectively. Note that point P_{kt} is on the $X_{L1} Y_{L1}$ plane; also it is on the $\underline{g} - \underline{n}_s$ plane. Then considering the two co-ordinate frames $O_{L1} - X_{L1} Y_{L1} Z_{L1}$ and $P_1 - \underline{g}\ \underline{n}_s\ \underline{e}$,

$$\begin{bmatrix} x_1 \\ y_1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{g}.X_{L1} & \underline{n}_s.X_{L1} & \underline{e}.X_{L1} & (\underline{r}_{i,1}^{(P_1)} - \underline{r}^{(O_{L1})}).X_{L1} \\ \underline{g}.Y_{L1} & \underline{n}_s.Y_{L1} & \underline{e}.Y_{L1} & (\underline{r}_{i,1}^{(P_1)} - \underline{r}^{(O_{L1})}).Y_{L1} \\ \underline{g}.Z_{L1} & \underline{n}_s.Z_{L1} & \underline{e}.Z_{L1} & (\underline{r}_{i,1}^{(P_1)} - \underline{r}^{(O_{L1})}).Z_{L1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^* \\ c_{kt} \\ 0 \\ 1 \end{bmatrix} \qquad (6.7)$$

where $\underline{r}_{i,1}^{(P_1)}$ and $\underline{r}^{(O_{L1})}$ are the position vectors of the points P_1 and O_{L1} respectively expressed in global co-ordinates. In the above equation, to avoid confusion, vector sign is not used with X_{L1}, Y_{L1} and Z_{L1} although these are unit direction vectors along the axes X_{L1}, Y_{L1} and Z_{L1} respectively expressed in global co-ordinates.

Now from the third row of Eqns. (6.7),

$$0 = (\underline{g}.Z_{L1})\, x^* + (\underline{n}_s.Z_{L1})\, c_{kt} + (\underline{r}_{i,1}^{(P_1)} - \underline{r}^{(O_{L1})}).Z_{L1} = 0.$$

But since the vector $(\underline{r}_{i,1}^{(P_1)} - \underline{r}^{(O_{L1})})$ is in the $X_{L1} Y_{L1}$ plane

$$(\underline{r}_{i,1}^{(P_1)} - \underline{r}^{(O_{L1})}).Z_{L1} = 0.$$

So

$$x^* = - \frac{\underline{n}_s \cdot Z_{L1}}{\underline{g} \cdot Z_{L1}} c_{kt} \quad . \tag{6.3}$$

Now considering the two co-ordinate frames O-XYZ and $P_1 - \underline{g}\ \underline{n}_s\ \underline{e}$

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{g}.X & \underline{n}_s.X & \underline{e}.X & \underline{r}_{i,1}^{(P_1)}.X \\ \underline{g}.Y & \underline{n}_s.Y & \underline{e}.Y & \underline{r}_{i,1}^{(P_1)}.Y \\ \underline{g}.Z & \underline{n}_s.Z & \underline{e}.Z & \underline{r}_{i,1}^{(P_1)}.Z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{\underline{n}_s.Z_{L1}}{\underline{g}.Z_{L1}} c_{kt} \\ c_{kt} \\ 0 \\ 1 \end{bmatrix} . \tag{6.9}$$

Hence the position vector of point P_{kt} is given by

$$\underline{r}_{i,kt}^{(P_{kt})} = \underline{r}_{i,1}^{(P_1)} - \frac{\underline{n}_s.Z_{L1}}{\underline{g}.Z_{L1}} c_{kt}\, \underline{g} + c_{kt}\, \underline{n}_s \quad . \tag{6.10}$$

Similarly consider a local co-ordinate frame $O_{L2} - X_{L2}\ Y_{L2}\ Z_{L2}$ where $X_{L2}\ Y_{L2}$ is the plane of the end surface E_2 and Z_{L2} axis is normal to it. Also another local co-ordinate frame $Q_1 - \underline{g}\ \underline{n}_s\ \underline{e}$ can be considered with the origin at Q_1. Then the position vector of the point Q_{kt} is given by

$$\underline{r}_{j,kt}^{(Q_{kt})} = \underline{r}_{j,1}^{(Q_1)} - \frac{\underline{n}_s.Z_{L2}}{\underline{g}.Z_{L2}} c_{kt}\, \underline{g} + c_{kt}\, \underline{n}_s \quad . \tag{6.11}$$

From Eqn. (6.10) it can be seen that the position vector of point P_{kt} is a function of the position

vector of point P_1, the vectors $\underline{g}$, $\underline{n}_s$ and Z_{L1} and c_{kt}. The vectors $\underline{r}_{i,1}^{(P_1)}$, $\underline{g}$ and $\underline{n}_s$ all have θ_i as the parameter. If the end surface E_1 is planar then Z_{L1} is a constant vector. Hence the vector $\underline{r}_{i,kt}^{(P_{kt})}$ has θ_i as its parameter. Similarly the vector $\underline{r}_{j,kt}^{(Q_{kt})}$ is a function of $\underline{r}_{j,1}^{(Q_1)}$, $\underline{g}$, $\underline{n}_s$, Z_{L2} and c_{kt} and since $\underline{r}_{j,1}^{(Q_1)}$, $\underline{g}$ and $\underline{n}_s$ are vectors of parameter θ_j and since Z_{L2} is constant if the end surface E_2 is planar one, the vector $\underline{r}_{j,kt}^{(Q_{kt})}$ has θ_j as its parameter.

Similarly corresponding to each generatrix of the mean surface the vectors $\underline{g}$, $\underline{n}_s$ and $\underline{e}$ can be obtained. Then the line of intersection of the $\underline{g}$ - $\underline{n}_s$ plane with the surface S_{kt} can be obtained. The points of intersection of this line with the end surfaces E_1 and E_2 can be obtained from Eqns. (6.10) and (6.11). Thus the lines of intersection of the surface S_{kt} with the end surfaces E_1 and E_2 can be obtained. The points P_{kt} and Q_{kt} are generic points on these lines respectively (refer to Figure 6.4). It can be seen that the lines $P_1 P_{kt}$ and $Q_1 Q_{kt}$ are the lines of intersection of the $\underline{g}$-$\underline{n}_s$ plane with the end surfaces E_1 and E_2 respectively.

6.2.4 Developability of the Set of Surfaces

Differentiating the vector $\underline{r}_{i,kt}^{(P_{kt})}$ with respect to θ_i,

$$\frac{d\underline{r}_{i,kt}^{(P_{kt})}}{d\theta_i} = \frac{d\underline{r}_{i,1}^{(P_1)}}{d\theta_i} - \frac{\underline{n}_s \cdot \underline{Z}_{L1}}{\underline{g} \cdot \underline{Z}_{L1}} c_{kt} \frac{d\underline{g}}{d\theta_i} + c_{kt} \frac{d\underline{n}_s}{d\theta_i}$$

$$- \frac{(\underline{g}\cdot\underline{Z}_{L1})\left(\frac{d\underline{n}_s}{d\theta_i}\cdot\underline{Z}_{L1}+\underline{n}_s\cdot\frac{d\underline{Z}_{L1}}{d\theta_i}\right)-(\underline{n}_s\cdot\underline{Z}_{L1})\left(\frac{d\underline{g}}{d\theta_i}\cdot\underline{Z}_{L1}+\underline{g}\cdot\frac{d\underline{Z}_{L1}}{d\theta_i}\right)}{(\underline{g}\cdot\underline{Z}_{L1})^2} c_{kt}\,\underline{g} \qquad \ldots \quad (6.12)$$

If $\underline{Z}_{L1}$ is a constant vector, then $\frac{d\underline{Z}_{L1}}{d\theta_i} = 0$ and Eqn. (6.12) reduces to

$$\frac{d\underline{r}_{i,kt}^{(P_{kt})}}{d\theta_i} = \frac{d\underline{r}_{i,1}^{(P_1)}}{d\theta_i} - \frac{\underline{n}_s \cdot \underline{Z}_{L1}}{\underline{g} \cdot \underline{Z}_{L1}} c_{kt} \frac{d\underline{g}}{d\theta_i} + c_{kt} \frac{d\underline{n}_s}{d\theta_i}$$

$$- \frac{(\underline{g}\cdot\underline{Z}_{L1})\left(\frac{d\underline{n}_s}{d\theta_i}\cdot\underline{Z}_{L1}\right)-(\underline{n}_s\cdot\underline{Z}_{L1})\left(\frac{d\underline{g}}{d\theta_i}\cdot\underline{Z}_{L1}\right)}{(\underline{g}\cdot\underline{Z}_{L1})^2} c_{kt}\,\underline{g} \qquad \ldots \quad (6.13)$$

The vectors $\frac{d\underline{r}_{i,1}^{(P_1)}}{d\theta_i}$, $\frac{d\underline{g}}{d\theta_i}$ and $\frac{d\underline{n}_s}{d\theta_i}$ are all in the plane $\underline{g} - \underline{e}$ (refer to Section 6.2.2). Hence the vector $d\underline{r}_{i,kt}^{(P_{kt})}/d\theta_i$ is acting in a plane parallel to the $\underline{g} - \underline{e}$ plane.

Similarly the derivative of the vector $\underline{r}_{j,kt}^{(Q_{kt})}$ with respect to the parameter θ_j is given by

$$\frac{d\underline{r}_{j,kt}^{(Q_{kt})}}{d\theta_j} = \frac{d\underline{r}_{j,1}^{(Q_1)}}{d\theta_j} - \frac{\underline{n}_s \cdot \underline{Z}_{L2}}{\underline{g} \cdot \underline{Z}_{L2}} c_{kt} \frac{d\underline{g}}{d\theta_j} + c_{kt} \frac{d\underline{n}_s}{d\theta_j}$$

$$- \frac{(\underline{g}\cdot\underline{Z}_{L2})\left(\frac{d\underline{n}_s}{d\theta_j}\cdot\underline{Z}_{L2}+\underline{n}_s\cdot\frac{d\underline{Z}_{L2}}{d\theta_j}\right)-(\underline{n}_s\cdot\underline{Z}_{L2})\left(\frac{d\underline{g}}{d\theta_j}\cdot\underline{Z}_{L2}+\underline{g}\cdot\frac{d\underline{Z}_{L2}}{d\theta_j}\right)}{(\underline{g}\cdot\underline{Z}_{L2})^2} c_{kt}\,\underline{g} \qquad \ldots \quad (6.14)$$

and if Z_{L2} is a constant vector (i.e. end surface E_2 is planar), then

$$\frac{d\underline{r}_{j,kt}^{(Q_{kt})}}{d\theta_j} = \frac{d\underline{r}_{j,1}^{(Q_1)}}{d\theta_j} - \frac{\underline{n}_s \cdot Z_{L2}}{\underline{g} \cdot Z_{L2}} c_{kt} \frac{d\underline{g}}{d\theta_j} + c_{kt} \frac{d\underline{n}_s}{d\theta_j}$$

$$- \frac{(\underline{g} \cdot Z_{L2})(\frac{d\underline{n}_s}{d\theta_j} \cdot Z_{L2}) - (\underline{n}_s \cdot Z_{L2})(\frac{d\underline{g}}{d\theta_j} \cdot Z_{L2})}{(\underline{g} \cdot ZL_2)^2} c_{kt}\underline{g} \quad \ldots \quad (6.15)$$

The vector $\frac{d\underline{r}_{j,kt}^{(Q_{kt})}}{d\theta_j}$ is also acting in the plane parallel to the $\underline{g}$ - $\underline{e}$ plane.

$\frac{d\underline{r}_{j,kt}^{(Q_{kt})}}{d\theta_j}$ The line $P_{kt}\ Q_{kt}$ and the vectors $\frac{d\underline{r}_{i,kt}^{(P_{kt})}}{d\theta_i}$ and are all in a plane parallel to the $\underline{g}$ - $\underline{e}$ plane. Thus these three vectors $(\underline{r}_{j,kt}^{(Q_{kt})} - \underline{r}_{i,kt}^{(P_{kt})})$, $d\underline{r}_{i,kt}^{(P_{kt})}/d\theta_i$ and $d\underline{r}_{j,kt}^{(Q_{kt})}/d\theta_j$ are co-planar. Since the vectors $\underline{r}_{i,kt}^{(P_{kt})}$ and $\underline{r}_{j,kt}^{(Q_{kt})}$ have respectively θ_i and θ_j as their parameters, the vectors $d\underline{r}_{i,kt}^{(P_{kt})} / d\theta_i$ and $d\underline{r}_{j,kt}^{(Q_{kt})} / d\theta_j$ are the tangents to the lines of intersection of the surface S_{kt} with the end surfaces E_1 and E_2. Hence the line $P_{kt}\ Q_{kt}$ can be considered as the generatrix of the surface S_{kt} and the condition for developability is satisfied. Thus this surface S_{kt} is a developable ruled surface. The lines of intersection of this surface with the end surfaces E_1 and E_2 are hence the primary and secondary directrices for this surface.

Similarly all the other surfaces in the set of surfaces can be shown to be developable ruled surfaces and the lines of intersection of these surfaces with the end surfaces E_1 and E_2 to be respectively the primary and secondary directrices of the concerned surface.

From the above-mentioned points, it can be seen that

(i) if the mean surface is a developable ruled surface, all other surfaces are also developable ruled surfaces,

(ii) if the lines of intersection of the mean surface with the end surfaces E_1 and E_2 are respectively the primary and secondary directrices for the mean surface then the lines of intersection of the other surfaces with the end surfaces E_1 and E_2 are respectively the primary and secondary directrices for the surface concerned,

(iii) generic points on these primary and secondary directrices are given by Eqns. (6.10) and (6.11) respectively. Ofcourse the following conditions are to be satisfied.

$$\underline{g} \cdot z_{L1} \neq 0$$

and (6.16)

$$\underline{g} \cdot z_{L2} \neq 0 .$$

These conditions imply that the generatrix does not lie on the end surfaces E_1 and E_2 respectively.

In case the generatrix is perpendicular to the end surface E_1, then $Z_{L1} = \underline{g}$ and hence Eqn. (6.10) reduces to

$$\underline{r}_{i,kt}^{(P_{kt})} = \underline{r}_{i,1}^{(P_1)} + c_{kt}\,\underline{n}_s \quad . \tag{6.17}$$

Similarly when the generatrix is perpendicular to the end surface E_2, $Z_{L2} = \underline{g}$ and

$$\underline{r}_{j,kt}^{(Q_{kt})} = \underline{r}_{j,1}^{(Q_1)} + c_{kt}\,\underline{n}_s \quad . \tag{6.18}$$

For a given position of the generic point of the primary directrix of the mean surface, the position of the generic points of the primary directrices of other surfaces is given by the line of intersection of the $\underline{g} - \underline{n}_s$ plane with the end surface E_1. Similarly the generic points on the secondary directrices of all the surfaces lie on the line of intersection of the $\underline{g} - \underline{n}_s$ plane with the end surface E_2.

(iv) the parameters for all the primary directrices are the same θ_i as for the primary directrix of the mean surface. Similarly all the secondary directrices have the same parameter θ_j.

Corresponding to a particular generatrix of the mean surface defined by a pair of (θ_i , θ_j) values, the generatrices of all other surfaces are also defined by the same pair of (θ_i , θ_j) values. Hence once the developability condition is satisfied for the mean surface and a set of (θ_i , θ_j) values fixed, the same set of (θ_i , θ_j) values satisfies the condition for developability for all other surfaces.

6.2.5 Arc Length

Eqns. (6.10) and (6.11) can be rewritten as

$$\underline{r}_{i,kt} = \underline{r}_{i,1} + c_{kt}\,\underline{v}_i \tag{6.19}$$

and

$$\underline{r}_{j,kt} = \underline{r}_{j,1} + c_{kt}\,\underline{v}_j \tag{6.20}$$

where

$$\underline{v}_i = -\frac{\underline{n}_s \cdot \underline{Z}_{L1}}{\underline{g} \cdot \underline{Z}_{L1}}\,\underline{g} + \underline{n}_s \tag{6.21}$$

and

$$\underline{v}_j = -\frac{\underline{n}_s \cdot \underline{Z}_{L2}}{\underline{g} \cdot \underline{Z}_{L2}}\,\underline{g} + \underline{n}_s\ . \tag{6.22}$$

The vectors $\underline{v}_i$ and $\underline{v}_j$ are not unit vectors. They are along the lines of intersection of the $\underline{g} - \underline{n}_s$ plane with the end surfaces E_1 and E_2 respectively.

The first derivative of $\underline{r}_{i,kt}$ with respect to

the parameter θ_i is

$$\dot{\underline{r}}_{i,kt} = \frac{d\underline{r}_{i,kt}}{d\theta_i}$$

$$= \frac{d\underline{r}_{i,1}}{d\theta_i} + c_{kt}\frac{d\underline{v}_i}{d\theta_i}$$

$$= \dot{\underline{r}}_{i,1} + c_{kt}\,\dot{\underline{v}}_i \qquad (6.23)$$

Here the dot above the vectors indicate their first derivative with respect to θ_i. The rate of variation of arc length of the primary directrix with respect to θ_i is

$$\dot{s}_{i,kt} = (\,\dot{\underline{r}}_{i,kt}\cdot\dot{\underline{r}}_{i,kt})^{1/2}$$

Substituting for $\dot{\underline{r}}_{i,kt}$ from Eqn. (6.23),

$$\dot{s}_{i,kt} = (\,\dot{\underline{r}}_{i,1}\cdot\dot{\underline{r}}_{i,1} + 2c_{kt}\,\dot{\underline{r}}_{i,1}\cdot\dot{\underline{v}}_i + c_{kt}^2\,\dot{\underline{v}}_i\cdot\dot{\underline{v}}_i)^{1/2} \qquad (6.24)$$

The arc length of the primary directrix is obtained by integrating the above expression. Note that for a given value of θ_i, $\dot{\underline{r}}_{i,1}$ and $\dot{\underline{v}}_i$ are constants and hence $s_{i,kt}$ is a function of c_{kt}.

6.2.6 Magnitude of Geodesic Curvature

The second derivative of $\underline{r}_{i,kt}$ with respect to θ_i is

$$\ddot{\underline{r}}_{i,kt} = \ddot{\underline{r}}_{i,1} + c_{kt}\,\ddot{\underline{v}}_i \qquad (6.25)$$

where the two dots above the vectors indicate their second derivative with respect to θ_i. The $k\underline{b}$ vector for the primary directrix is given by

$$k\underline{b}_{i,kt} = \frac{\dot{\underline{r}}_{i,kt} \times \ddot{\underline{r}}_{i,kt}}{\dot{s}_{i,kt}^{\,3}}$$

and substituting for $\dot{\underline{r}}_{i,kt}$, $\ddot{\underline{r}}_{i,kt}$ and $\dot{\underline{s}}_{i,kt}$

$$k\underline{b}_{i,kt} = \frac{\dot{\underline{r}}_{i,1} \times \ddot{\underline{r}}_{i,1} + c_{kt}(\dot{\underline{r}}_{i,1} \times \ddot{\underline{v}}_i + \dot{\underline{v}}_i \times \ddot{\underline{r}}_{i,1}) + c_{kt}^2\, \dot{\underline{v}}_i \times \ddot{\underline{v}}_i}{(\dot{\underline{r}}_{i,1}\cdot\dot{\underline{r}}_{i,1} + 2c_{kt}\, \dot{\underline{r}}_{i,1}\cdot\dot{\underline{v}}_i + c_{kt}^2\, \dot{\underline{v}}_i\cdot\dot{\underline{v}}_i)^{3/2}} \qquad \text{....} \qquad (6.26)$$

The magnitude of geodesic curvature is $k_{g_{i,kt}} = k\underline{b}_{i,kt}\cdot\underline{n}_s$. For a given value of θ_i, the curvature vector $k\underline{b}_{i,kt}$ and hence the magnitude of geodesic curvature are functions of c_{kt}.

6.2.7 Development of the Individual Surfaces

From the Eqns. (6.19) to (6.26) it can be seen that all the expressions required for the development of the individual surfaces can be obtained once the vectors $\underline{r}_{i,1}$, $\underline{r}_{j,1}$, $\underline{v}_i$ and $\underline{v}_j$ are known. Hence the development of the mean surface is first to be carried out and the vectors $\underline{r}_{i,1}$, $\underline{r}_{j,1}$, $\underline{v}_i$ and $\underline{v}_j$ are to be calculated for various values of θ_i. The development of other surfaces can then be accomplished knowing the above vectors and the distance c_{kt} between the mean

surface and the surface S_{kt} under consideration. Note that all these surfaces (including the mean surface) are developable ruled surfaces with two directrices and that the development of these surfaces can be carried out as indicated in Section 2.6, with a modification that the condition for developability is satisfied for the mean surface only and the set of (θ_i , θ_j) values are found which satisfy the condition for developability for other surfaces also.

6.3 Stacking of the Developments of Individual Surfaces

Stacking of developments of individual surfaces is carried out by taking the development of the mean surface as reference and stacking the other developments with respect to it. The co-ordinates of the generic points on the development of primary and secondary directrices of other surfaces are expressed in terms of the co-ordinate axes of the development of the mean surface.

Consider a co-ordinate frame $O_{d_{kt}} - X_{d_{kt}} Y_{d_{kt}} Z_{d_{kt}}$ where $O_{d_{kt}}$ is the origin of the development of the surface S_{kt} and $X_{d_{kt}}$ and $Y_{d_{kt}}$ are the x and y axes of the development. The $Z_{d_{kt}}$ axis is perpendicular to the plane of development. Let $\underline{t}_{O_{kt}}$, $\underline{e}_{O_{kt}}$ & $\underline{n}_{O_{kt}}$ be the unit vectors along the $X_{d_{kt}}$, $Y_{d_{kt}}$ and $Z_{d_{kt}}$ axes. Here $\underline{t}_{O_{kt}}$ is the unit tangent vector to the primary directrix of

the surface S_{kt} at the starting point and $\underline{n}_{o_{kt}}$ is the unit normal to the surface at that point. The vector $\underline{e}_{o_{kt}}$ is given by

$$\underline{e}_{o_{kt}} = \underline{n}_{o_{kt}} \times \underline{t}_{o_{kt}} . \qquad (6.27)$$

Let the co-ordinate frame for the development of the mean surface be $O_{d_1} - X_{d_1} Y_{d_1} Z_{d_1}$ and the corresponding unit vectors be $\underline{t}_{o_1}$, $\underline{e}_{o_1}$ and $\underline{n}_{o_1}$. Then the co-ordinates of the development of the surface S_{kt}, $[x_{d_{kt}}, y_{d_{kt}}, 0, 1]$ can be expressed with respect to the co-ordinate frame $O_{d_1} - X_{d_1} Y_{d_1} Z_{d_1}$ as

$$\begin{bmatrix} x_{dm_{kt}} \\ y_{dm_{kt}} \\ z_{dm_{kt}} \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{t}_{o_{kt}} \cdot \underline{t}_{o_1} & \underline{e}_{o_{kt}} \cdot \underline{t}_{o_1} & \underline{n}_{o_{kt}} \cdot \underline{t}_{o_1} & (\underline{r}^{(O_{d_{kt}})} - \underline{r}^{(O_{d_1})}) \cdot \underline{t}_{o_1} \\ \underline{t}_{o_{kt}} \cdot \underline{e}_{o_1} & \underline{e}_{o_{kt}} \cdot \underline{e}_{o_1} & \underline{n}_{o_{kt}} \cdot \underline{e}_{o_1} & (\underline{r}^{(O_{d_{kt}})} - \underline{r}^{(O_{d_1})}) \cdot \underline{e}_{o_1} \\ \underline{t}_{o_{kt}} \cdot \underline{n}_{o_1} & \underline{e}_{o_{kt}} \cdot \underline{n}_{o_1} & \underline{n}_{o_{kt}} \cdot \underline{n}_{o_1} & (\underline{r}^{(O_{d_{kt}})} - \underline{r}^{(O_{d_1})}) \cdot \underline{n}_{o_1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{d_{kt}} \\ y_{d_{kt}} \\ 0 \\ 1 \end{bmatrix}$$

.... (6.28)

where $\underline{r}^{(O_{d_{kt}})}$ and $\underline{r}^{(O_{d_1})}$ are the position vectors of the points $O_{d_{kt}}$ and O_{d_1}, which are nothing but the generic points on the primary directrices of the surface S_{kt} and the mean surface at the starting point. From Eqn. (6.10),

$$\underline{r}^{(O_{d_{kt}})} - \underline{r}^{(O_{d_1})} = -\frac{\underline{n}_s \cdot \underline{Z}_{L1}}{\underline{g} \cdot \underline{Z}_{L1}} c_{kt}\underline{g} + c_{kt}\underline{n}_s \qquad (6.29)$$

where the vector $\underline{n}_s$ is nothing but the vector $\underline{n}_{O_1}$. Let the vector $\underline{g}$ in the Eqn. (6.29) be written as $\underline{g}_{O_1}$. Since all the surfaces are parallel surfaces, $\underline{n}_{O_{kt}} = \underline{n}_{O_1}$. Then Eqns. (6.28) reduce to

$$\begin{bmatrix} x_{dm_{kt}} \\ y_{dm_{kt}} \\ z_{dm_{kt}} \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{t}_{O_{kt}} \cdot \underline{t}_{O_1} & \underline{e}_{O_{kt}} \cdot \underline{t}_{O_1} & 0 & -\dfrac{\underline{n}_{O_1} \cdot \underline{Z}_{L1}}{\underline{g}_{O_1} \cdot \underline{Z}_{L1}} c_{kt}\, \underline{g}_{O_1} \cdot \underline{t}_{O_1} \\ \underline{t}_{O_{kt}} \cdot \underline{e}_{O_1} & \underline{e}_{O_{kt}} \cdot \underline{e}_{O_1} & 0 & -\dfrac{\underline{n}_{O_1} \cdot \underline{Z}_{L1}}{\underline{g}_{O_1} \cdot \underline{Z}_{L1}} c_{kt}\, \underline{g}_{O_1} \cdot \underline{e}_{O_1} \\ 0 & 0 & 1 & c_{kt} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{d_{kt}} \\ y_{d_{kt}} \\ 0 \\ 1 \end{bmatrix} \qquad \ldots \quad (6.30)$$

From the above equations it can be seen that $z_{dm_{kt}} = c_{kt}$ always. This is because the surface S_{kt} and the mean surface are parallel and are at a distance c_{kt} from one another. So not only the two surfaces but also their developments are parallel and are at a distance c_{kt} from one another.

The developments of all the surfaces are stacked with respect to the development of the mean surface using Eqn. (6.30). These are parallel to one another and their distance from the development of the mean surface is the same as the distance of the respective surface from the mean surface.

6.4 Algorithm for the Development of Thick Surface

The algorithm for the development of a thick surface is given here. The primary and secondary directrices for the mean surface are not defined here specifically. The end surfaces E_1 and E_2 on which the primary and secondary directrices of the set of surfaces lie are assumed to be planar surfaces.

Step 1: (a) The location and orientation of the two end surfaces E_1 and E_2 are specified, i.e. the local co-ordinate frames $O_{L1} - X_{L1} Y_{L1} Z_{L1}$ and $O_{L2} - X_{L2} Y_{L2} Z_{L2}$ are specified.

(b) The primary and secondary directrices of the mean surface are specified.

(c) The thickness of the thick surface, h, and the number of slices into which each half of the thick surface is to be divided, s_h, are assumed to be given as input parameters.

(d) The number of positions of the generic point of the primary directrix, N, is given.

Step 2: Calculate

c_{kt} , $kt = 1, 2, \ldots, (2s_h + 1)$

(refer Sec. 6.1).

Step 3: The development of mean surface is to be carried out first. So $kt = 1$. First set $K = 1$.

Step 4: Fix the value of $\theta_i(K)$. From the condition for developability of the mean surface find the corresponding value of $\theta_j(K)$. Calculate the vectors $\underline{r}_{i,1}(K)$, $\underline{r}_{j,1}(K)$, $\underline{g}(K)$ and $\underline{n}_s(K)$. Calculate the vectors $\underline{v}_i(K)$ and $\underline{v}_j(K)$ from Eqns.(6.21) and (6.22).

Step 5: Calculate the arc length of primary directrix, and the magnitude of geodesic curvature. Calculate the co-ordinates of the ends of the generatrix in the development of the mean surface. Let them be $[x_{d_{i,1}}(K), y_{d_{i,1}}(K)]$ and $[x_{d_{j,1}}(K), y_{d_{j,1}}(K)]$.

Step 6: If $K > 1$, go to step 7. If $K = 1$, note down the values of the vectors $\underline{n}_{0_1}$, $\underline{g}_{0_1}$ and $\underline{t}_{0_1}$. Also calculate the vector $\underline{e}_{0_1}$.

Step 7: If $K = N$, go to step 8. Otherwise set $K = K + 1$ and go to step 4.

Step 8: Calculate the first and second derivatives of the vectors $v_i(K)$, $K = 1$ to N, with respect the parameter θ_i.

Step 9: The development of other surfaces is to be carried out now. Set $K = 1$.

Step 10: Set $kt = 2$

Step 11: Calculate the vectors $\underline{r}_{i,kt}(K)$ and $\underline{r}_{j,kt}(K)$ from Eqns. (6.19) and (6.20). The vectors $\underline{r}_{i,1}(K)$ and $\underline{r}_{j,1}(K)$ and the distance c_{kt} are known.

Step 12: Calculate the tangent vector $\dot{\underline{r}}_{i,kt}$ from Eqn. (6.23) and from this calculate the unit tangent vector to the primary directrix (of the surface S_{kt}). Also calculate the vector $\ddot{\underline{r}}_{i,kt}$ (Eqn. (6.25)).

Step 13: Calculate the arc length and the magnitude of geodesic curvature using Eqns. (6.24) and (6.26) respectively. Then calculate the co-ordinates of the ends of the generatrix under consideration in the development of the surface S_{kt} i.e. $[x_{d_{i,kt}}(K),\ y_{d_{i,kt}}(K)]$ and $[x_{d_{j,kt}}(K),\ y_{d_{j,kt}}(K)]$.

Step 14: If $K > 1$, go to Step 15. If $K = 1$, note down the value of $\underline{t}_{O_{kt}}$. Also calculate the value of $\underline{e}_{O_{kt}}$.

Step 15: If $kt = 2s_h+1$, go to Step 16. Otherwise set $kt = kt+1$ and go to Step 11.

Step 16: If $K = N$, go to Step 17. Otherwise set $K = K+1$ and go to Step 10.

Step 17: Calculate

$$m_1 = -\frac{\underline{n}_{O_1} \cdot Z_{L1}}{\underline{g}_{O_1} \cdot Z_{L1}} \underline{g}_{O_1} \cdot \underline{t}_{O_1}$$

$$m_2 = -\frac{\underline{n}_{O_1} \cdot Z_{L1}}{\underline{g}_{O_1} \cdot Z_{L1}} \underline{g}_{O_1} \cdot \underline{e}_{O_1}$$

Step 18: Set $kt = 2$

Step 19: Calculate

$$m_3 = \underline{t}_{O_{kt}} \cdot \underline{t}_{O_1}$$

$$m_4 = \underline{e}_{O_{kt}} \cdot \underline{t}_{O_1}$$

$$m_5 = m_1 \ c_{kt}$$

$$m_6 = \underline{t}_{O_{kt}} \cdot \underline{e}_{O_1}$$

$$m_7 = \underline{e}_{O_{kt}} \cdot \underline{e}_{O_1}$$

$$m_8 = m_2 \ c_{kt}$$

Step 20: Set $K = 1$

Step 21: Calculate

$$x_{dm_{i,kt}}^{(K)} = m_3 \ x_{d_{i,kt}}^{(K)} + m_4 \ y_{d_{i,kt}}^{(K)} + m_5$$

$$y_{dm_{i,kt}}^{(K)} = m_6 \ x_{d_{i,kt}}^{(K)} + m_7 \ y_{d_{i,kt}}^{(K)} + m_8$$

Contd....

$$x_{dm_{j,kt}}^{(K)} = m_3\, x_{d_{j,kt}}^{(K)} + m_4\, y_{d_{j,kt}}^{(K)} + m_5$$

$$y_{dm_{j,kt}}^{(K)} = m_6\, x_{d_{j,kt}}^{(K)} + m_7\, y_{d_{j,kt}}^{(K)} + m_8$$

Step 22: If $K = N$, go to Step 23. Otherwise set $K = K+1$ and go to Step 21.

Step 23: If $kt = (2s_h+1)$, go to Step 24. Otherwise set $kt = kt + 1$ and go to Step 19.

Step 24: Set for $K = 1$ to N

$$x_{dm_{i,1}}^{(K)} = x_{d_{i,1}}^{(K)}$$

$$y_{dm_{i,1}}^{(K)} = y_{d_{i,1}}^{(K)}$$

$$x_{dm_{j,1}}^{(K)} = x_{d_{j,1}}^{(K)}$$

$$y_{dm_{j,1}}^{(K)} = y_{d_{j,1}}^{(K)}$$

6.5 Equally Thick Surfaces in Series

Consider two equally thick surfaces TS_1 and TS_2. Let (refer to Figure 6.5)

(i) each one of them have a mean surface, called $S_{1,1}$ and $S_{2,1}$ respectively, which is a developable ruled surface. The first subscript corresponds to the thick surface and the second one indicates the mean surface (refer to Section 6.2)

(ii) each of the thick surface be considered as a set of equal number of surfaces

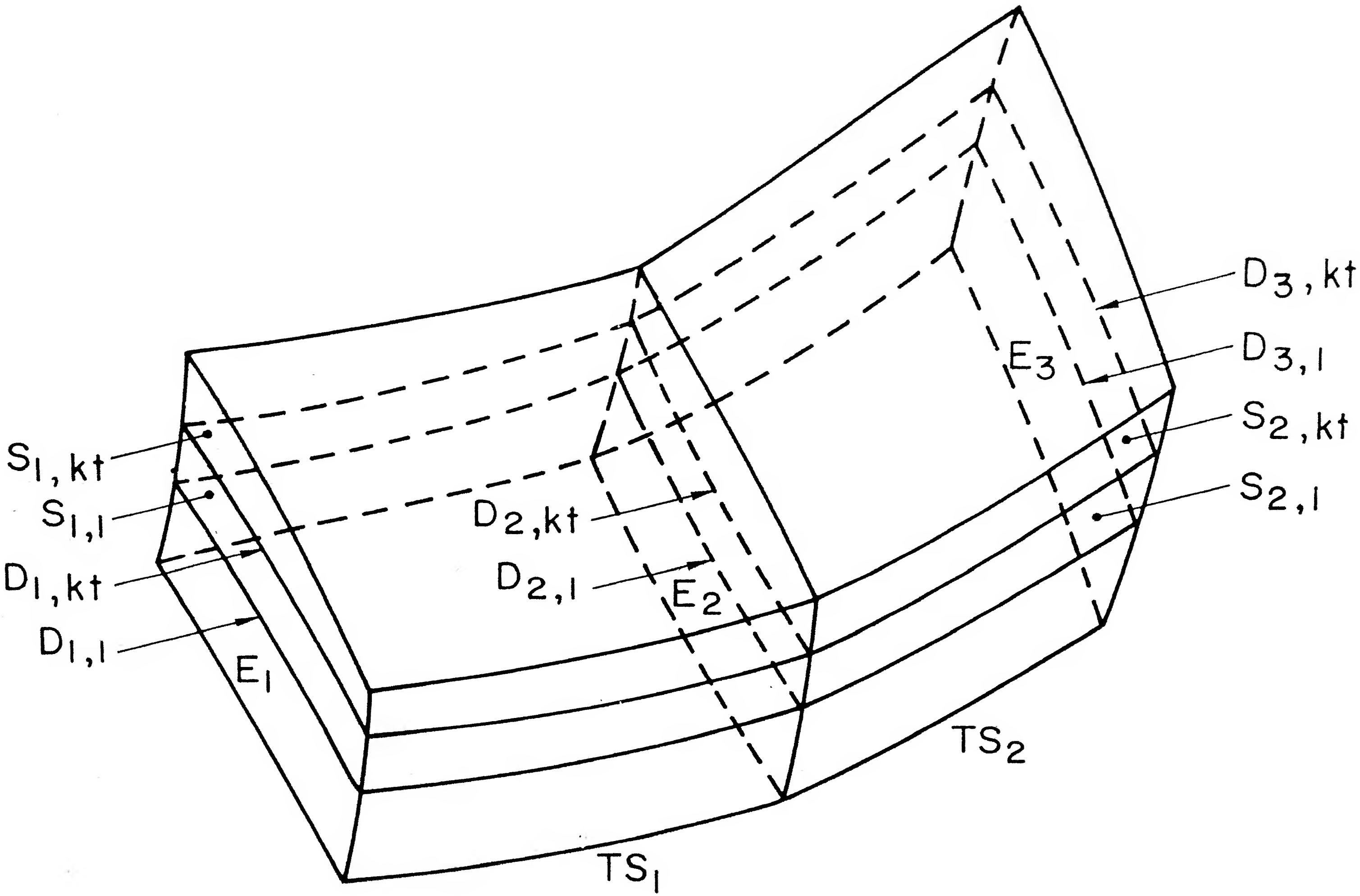


Fig. 6.5 Equally thick surfaces in series.

(iii) corresponding to each surface in one set, there be a surface in the other set such that the distance between the surface and the relevant mean surface is the same. For example if there is a surface $S_{1,kt}$ at a distance c_{kt} from the mean surface $S_{1,1}$ in the thick surface TS_1 then there is a surface $S_{2,kt}$ at a distance c_{kt} from the mean surface $S_{2,1}$ in the thick surface TS_2.

(iv) these corresponding surfaces of the two sets meet along a line. For example the surfaces $S_{1,kt}$ and $S_{2,kt}$ meet along a line $D_{2,kt}$ Similarly the top surfaces of the two sets meet along a line, the bottom surfaces along another line, the mean surfaces along $D_{2,1}$, etc.

(v) the surface containing these lines be the common end surface for the two thick surfaces TS_1 and TS_2. Let it be called E_2.

Suppose the line $D_{2,1}$ can be considered as one of the directrices for both the mean surfaces $S_{1,1}$ and $S_{2,1}$. Consider the line $D_{2,1}$ to be the secondary directrix for the mean surface $S_{1,1}$ and to be the primary directrix for the mean surface $S_{2,1}$. Let the line of intersection of the surface $S_{1,1}$ with the end surface E_1 be the primary directrix for $S_{1,1}$. Similarly the secondary directrix for the surface $S_{2,1}$ is the line of

intersection of $S_{2,1}$ with the end surface E_3. Let these lines be called $D_{1,1}$ and $D_{3,1}$ respectively. Let θ_1, θ_2 and θ_3 be the parameters of the lines $D_{1,1}$, $D_{2,1}$ and $D_{3,1}$ respectively.

6.5.1 Point on the Common End Surface

Let $P_1 Q_1$ be a generatrix of the mean surface $S_{1,1}$ (refer to Figure 6.6). Let

(i) $\underline{g}_1$ be the unit direction vector along $P_1 Q_1$,

(ii) $\underline{n}_{s_1}$ be the unit normal vector to the mean surface $S_{1,1}$ and

(iii) $\underline{e}_1$ be the unit vector given by $\underline{g}_1 \times \underline{n}_{s_1}$.

Consider a local co-ordinate frame $Q_1 - \underline{g}_1 \underline{n}_{s_1} \underline{e}_1$. Let $P_{kt} Q_{kt}$ be the line of intersection of the $\underline{g}_1 - \underline{n}_{s_1}$ plane with the surface $S_{1,kt}$ which is at a distance c_{kt} from the mean surface $S_{1,1}$. P_{kt} is the point on the end surface E_1 and Q_{kt} lies on the end surface E_2. Let $(x_1, c_{kt}, 0, 1)$ be the co-ordinates of the point Q_{kt} expressed in terms of the local co-ordinate frame $Q_1 - \underline{g}_1 \underline{n}_{s_1} \underline{e}_1$. Then

$$\underline{r}_{2,kt}^{(Q_{kt})} = \underline{r}_{2,1}^{(Q_1)} + x_1 \underline{g}_1 + c_{kt} \underline{n}_{s_1} \qquad (6.31)$$

where

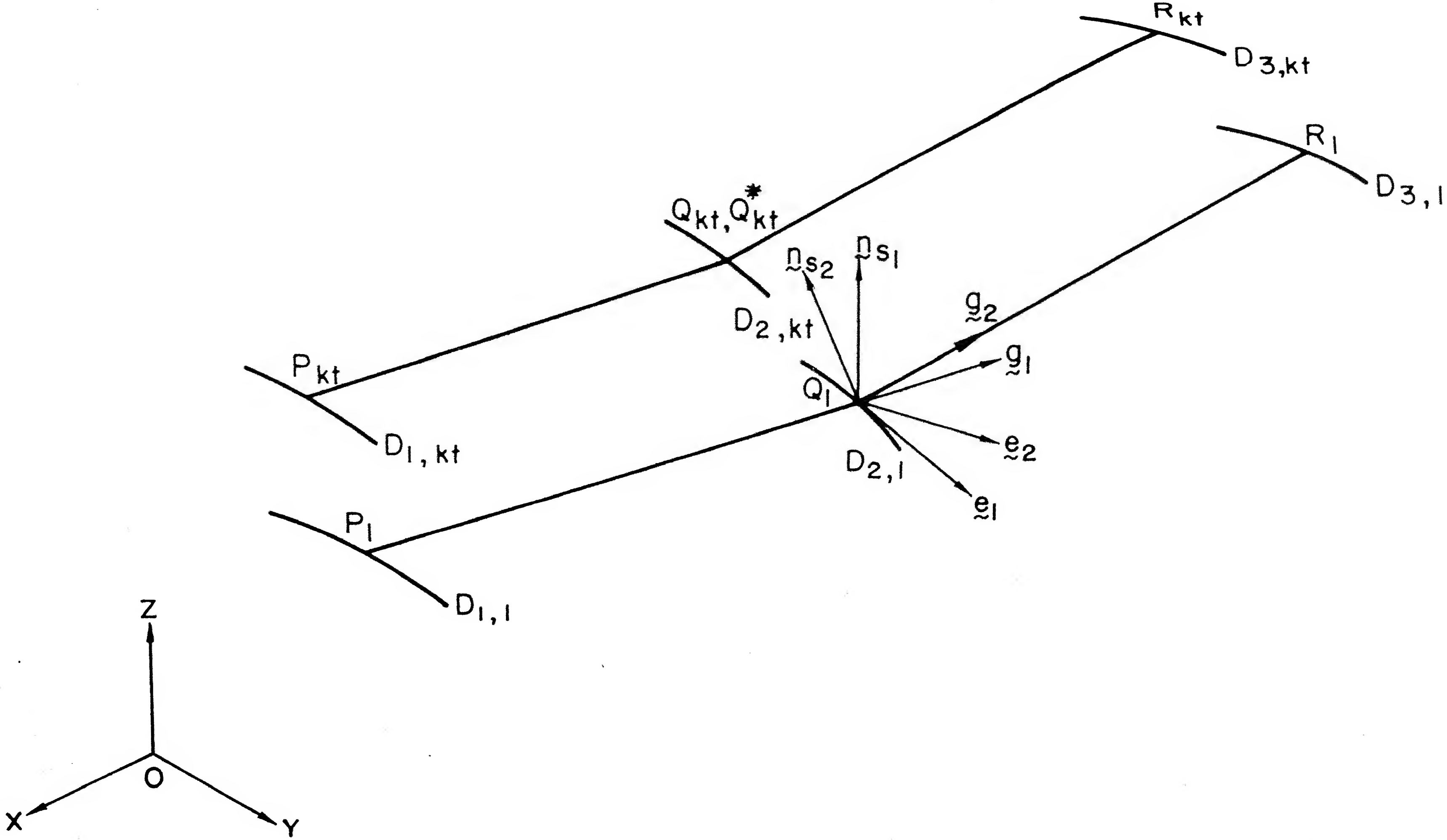


Fig. 6.6 Point on the common end surface.

$\underline{r}_{2,1}^{(Q_1)}$ - position vector of the generic point Q_1 on the line $D_{2,1}$ and

$\underline{r}_{2,kt}^{(Q_{kt})}$ - position vector of the point Q_{kt} on the line $D_{2,kt}$, the line of intersection of the surface $S_{1,kt}$ with the end surface E_2.

The vector $\underline{r}_{2,1}^{(Q_1)}$, has θ_2 as its parameter. The vectors $\underline{g}_1$, $\underline{n}_{s_1}$ and $\underline{e}_1$ have θ_1 or θ_2 as their parameter since θ_1 and θ_2 are interrelated through the condition for developability of the surface $S_{1,1}$.

Let $Q_1 R_1$ be a generatrix of the mean surface $S_{2,1}$ and let

(i) $\underline{g}_2$ be the unit direction vector along the generatrix $Q_1 R_1$,

(ii) $\underline{n}_{s_2}$ be the unit normal vector to the mean surface $S_{2,1}$ and

(iii) $\underline{e}_2$ be the unit vector given by $\underline{g}_2 \times \underline{n}_{s_2}$.

Consider a local co-ordinate frame $Q_1 - \underline{g}_2\, \underline{n}_{s_2}\, \underline{e}_2$.
Let $Q^*_{kt}R_{kt}$ be the line of intersection of $\underline{g}_2 - \underline{n}_{s_2}$ plane with the surface $S_{2,kt}$ which is at a distance c_{kt} from the mean surface $S_{2,1}$. The point Q^*_{kt} is on the end surface E_2 and the point R_{kt} on the end surface E_3. Let $(x_2, c_{kt}, 0, 1)$ be the co-ordinates of Q^*_{kt} expressed in terms of the local co-ordinate frame

$Q_1 - g_2\, n_{s_2}\, e_2$. Then the position vector of Q^*_{kt} is given by

$$\underline{r}_{2,kt}^{(Q^*_{kt})} = \underline{r}_{2,1}^{(Q_1)} + x_2\, \underline{g}_2 + c_{kt}\, \underline{n}_{s_2} \,. \qquad (6.32)$$

The vectors $\underline{g}_2$, $\underline{n}_{s_2}$ and $\underline{e}_2$ have θ_2 or θ_3 as their parameter since θ_2 and θ_3 are interrelated through the condition for developability of the surface $S_{2,1}$. The point Q^*_{kt} lies on the line of intersection of the surface $S_{2,kt}$ with the end surface E_2. This line is nothing but the line $D_{2,kt}$.

Corresponding to the generic point Q_1 on the line $D_{2,1}$, the point on the line $D_{2,kt}$ is Q_{kt} when the surface TS_1 is considered and is Q^*_{kt} when the surface TS_2 is considered. The line $D_{2,1}$ is the secondary directrix for the mean surface $S_{1,1}$ of thick surface TS_1. So the lines of intersection of the set of surfaces of the thick surface TS_1 with the end surface E_2 are the secondary directrices of the respective surfaces (refer to Section 6.2.4). Similarly the lines of intersection of all the surfaces of the thick surface TS_2 with the end surface E_2 are the primary directrices of the respective surfaces. So the line $D_{2,kt}$ is the secondary directrix for surface $S_{1,kt}$ and the primary directrix for $S_{2,kt}$. Hence corresponding to the generic point

Q_1 on $D_{2,1}$ the generic point on $D_{2,kt}$ is fixed whether these two lines $D_{2,1}$ and $D_{2,kt}$ are considered as primary directrices for the relevant surfaces in TS_2 or as secondary directrices for the relevant surfaces in TS_1. So the points Q_{kt} and Q^*_{kt} are one and the same.

Hence

$$\begin{bmatrix} x_2 \\ c_{kt} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{g}_1\cdot\underline{g}_2 & \underline{n}_{s_1}\cdot\underline{g}_2 & \underline{e}_1\cdot\underline{g}_2 & 0 \\ \underline{g}_1\cdot\underline{n}_{s_2} & \underline{n}_{s_1}\cdot\underline{n}_{s_2} & \underline{e}_1\cdot\underline{n}_{s_2} & 0 \\ \underline{g}_1\cdot\underline{e}_2 & \underline{n}_{s_1}\cdot\underline{e}_2 & \underline{e}_1\cdot\underline{e}_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ c_{kt} \\ 0 \\ 1 \end{bmatrix} \quad \ldots \quad (6.33)$$

From the third row

$$0 = (\underline{g}_1\cdot\underline{e}_2)\, x_1 + (\underline{n}_{s_1}\cdot\underline{e}_2)\, c_{kt}$$

So

$$x_1 = \frac{-\underline{n}_{s_1}\cdot\underline{e}_2}{\underline{g}_1\cdot\underline{e}_2}\, c_{kt} \qquad (6.34)$$

Then Eqn. (6.31) is rewritten as

$$\underline{r}_{2,kt}^{(Q_{kt})} = \underline{r}_{2,1}^{(Q_1)} - \frac{\underline{n}_{s_1}\cdot\underline{e}_2}{\underline{g}_1\cdot\underline{e}_2}\, c_{kt}\, \underline{g}_1 + c_{kt}\, \underline{n}_{s_1} \quad \ldots \quad (6.35)$$

This equation can be compared with Eqn. (6.11). Then it can be seen that the vector $\underline{e}_2$, which is normal to the plane $\underline{g}_2 - \underline{n}_{s_2}$ has replaced the vector Z_{L2} in Eqn. (6.11). This is because the point Q_{kt} is obtained as

the point of intersection of the two lines, the line $P_{kt}\ Q_{kt}$, lying in the $\underline{g}_1 - \underline{n}_{s_1}$ plane and at a distance c_{kt} from $P_1\ Q_1$, and the line $Q^*_{kt}\ R_{kt}$, lying in the $\underline{g}_2 - \underline{n}_{s_2}$ plane and at a distance c_{kt} from $Q_1\ R_1$. Thus it lies on the line of intersection of the two planes $\underline{g}_1 - \underline{n}_{s_1}$ and $\underline{g}_2 - \underline{n}_{s_2}$. Since the vectors $\underline{g}_1$, $\underline{n}_{s_1}$ and $\underline{e}_2$ have θ_2 as parameter, the vector $\underline{r}^{(Q_{kt})}_{2,kt}$ has θ_2 as its parameter.

Similarly x_2 in Eqn. (6.32) can be expressed as

$$x_2 = -\frac{\underline{n}_{s_2} \cdot \underline{e}_1}{\underline{g}_2 \cdot \underline{e}_1}\, c_{kt} \qquad (6.36)$$

and Eqn. (6.32) can be written as

$$\underline{r}^{(Q_{kt})}_{2,kt} = \underline{r}^{(Q_1)}_{2,1} - \frac{\underline{n}_{s_2} \cdot \underline{e}_1}{\underline{g}_2 \cdot \underline{e}_1}\, c_{kt}\, \underline{g}_2 + c_{kt}\, \underline{n}_{s_2} \qquad (6.37)$$

Comparing this equation with Eqn. (6.10), it can be seen that the vector $\underline{e}_1$ has replaced the vector Z_{L1}. Thus the $\underline{g}_1 - \underline{n}_{s_1}$ plane acts as the surface which the plane $\underline{g}_2 - \underline{n}_{s_2}$ intersect to get the point Q_{kt}. Here again since the vectors $\underline{n}_{s_2}$, $\underline{g}_2$ and $\underline{e}_1$ can be considered to have θ_2 as their parameter, the vector $\underline{r}^{(Q_{kt})}_{2,kt}$ has θ_2 as its parameter. Any one of the two equations, Eqn. (6.35) or Eqn. (6.37) can be used to find $\underline{r}^{(Q_{kt})}_{2,kt}$.

The points Q_1 and Q_{kt} lie on the line of intersection of the planes $\underline{g}_1 - \underline{n}_{s_1}$ and $\underline{g}_2 - \underline{n}_{s_2}$.

6.5.2 Conditions for Intersection of $\underline{g}$-$\underline{n}_s$ planes

The condition to be satisfied by Eqn. (6.35) is

$$\underline{g}_1 \cdot \underline{e}_2 \neq 0. \qquad (6.38)$$

Then only the point Q_{kt} can be located. Since $\underline{e}_2 = \underline{g}_2 \times \underline{n}_{s_2}$, the above condition implies that

$$\underline{g}_1 \cdot (\underline{g}_2 \times \underline{n}_{s_2}) \neq 0 \qquad (6.39)$$

Hence

(i) the three vectors $\underline{g}_1$, $\underline{g}_2$ and $\underline{n}_{s_2}$ should not be co-planar and

(ii) any two of these vectors should not be equal.

Since $\underline{g}_2$ and $\underline{n}_{s_2}$ are perpendicular to each other, the second condition means that $\underline{g}_1 \neq \underline{g}_2$ or $\underline{g}_1 \neq \underline{n}_{s_2}$. This again implies that the vector $\underline{g}_1$ should not be in the $\underline{g}_2 - \underline{n}_{s_2}$ plane.

An exception to this condition is when the dot product $\underline{n}_{s_1} \cdot \underline{e}_2$ is also equal to zero. Then all the four vectors $\underline{g}_1$, $\underline{n}_{s_1}$, $\underline{g}_2$ and $\underline{n}_{s_2}$ are co-planar (refer to Figure 6.7a); the vectors $\underline{e}_1$ and $\underline{e}_2$ are equal. The tangent vector at Q_1 is along $\underline{e}_1$. Then the third row of Eqns. (6.33) is of no use. From the second row of the Eqns. (6.33),

$$c_{kt} = (\underline{g}_1 \cdot \underline{n}_{s_2}) x_1 + (\underline{n}_{s_1} \cdot \underline{n}_{s_2}) c_{kt}$$

So

$$x_1 = \frac{1 - \underline{n}_{s_1} \cdot \underline{n}_{s_2}}{\underline{g}_1 \cdot \underline{n}_{s_2}} c_{kt} \tag{6.40}$$

Then the position vector of the point Q_{kt} is given by

$$\underline{r}_{2,kt}^{(Q_{kt})} = \underline{r}_{2,1}^{(Q_1)} + \frac{1 - \underline{n}_{s_1} \cdot \underline{n}_{s_2}}{\underline{g}_1 \cdot \underline{n}_{s_2}} c_{kt} \, \underline{g}_1 + c_{kt} \, \underline{n}_{s_1}$$

$$\ldots \tag{6.41}$$

Here also the following condition is to be satisfied.

$$\underline{g}_1 \cdot \underline{n}_{s_2} \neq 0 \quad . \tag{6.42}$$

But if $\underline{g}_1 \cdot \underline{n}_{s_2} = 0$, then the vectors $\underline{g}_1$ and $\underline{g}_2$ are the same. Then the vectors $\underline{n}_{s_1}$ and $\underline{n}_{s_2}$ are the same since the tangent vector at Q_1 is common for both the surfaces $S_{1,1}$ and $S_{2,1}$. Then $1 - \underline{n}_{s_1} \cdot \underline{n}_{s_2} = 0$ and the vector $\underline{r}_{2,kt}^{(Q_{kt})}$ is given by

$$\underline{r}_{2,kt}^{(Q_{kt})} = \underline{r}_{2,1}^{(Q_1)} + c_{kt} \, \underline{n}_{s_1} \tag{6.43}$$

The other situation of $\underline{g}_1$ being along $\pm \underline{n}_{s_2}$ and $\underline{n}_{s_1}$ acting in the $\underline{g}_2 - \underline{n}_{s_2}$ plane can occur (refer to Figure 6.7b). Then

$$\underline{r}_{2,kt}^{(Q_{kt})} = \underline{r}_{2,1}^{(Q_1)} \pm c_{kt} \, \underline{g}_1 + c_{kt} \, \underline{n}_{s_1} \tag{6.44}$$

depending upon $\underline{g}_1 = \pm \underline{n}_{s_2}$.

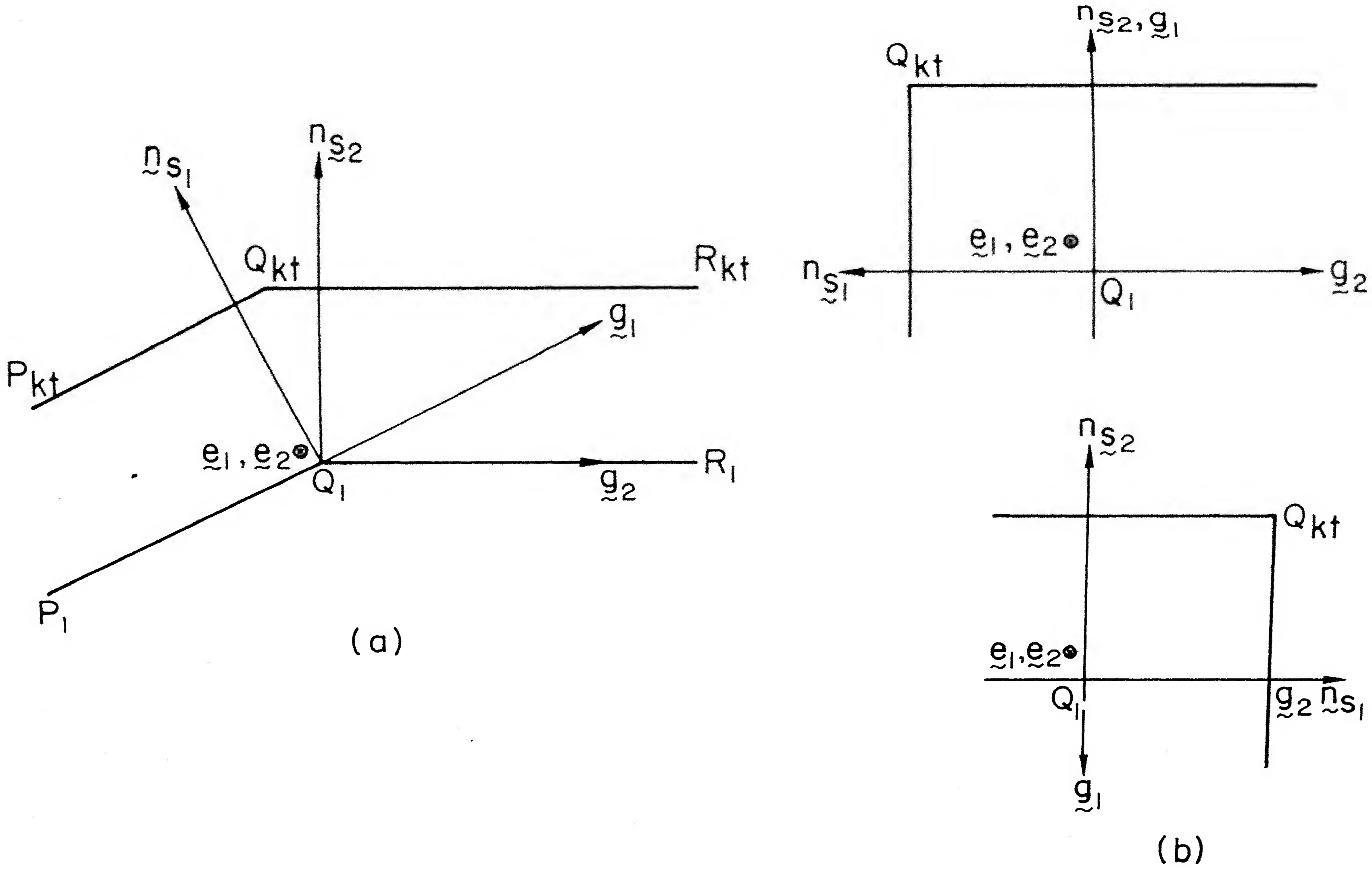


Fig. 6.7 Vectors g_1, n_{s_1}, g_2 and n_{s_2} are co-planar .

Similarly the condition to be satisfied by Eqn. (6.36) and the exception to it can be obtained by interchanging the subscripts 1 and 2 in Eqns. (6.38) to (6.44).

6.5.3 Developability of Other Surfaces

The first derivative of $\underline{r}_{2,kt}^{(Q_{kt})}$ with respect to the parameter θ_2 is obtained from Eqn. (6.35) as

$$\dot{\underline{r}}_{2,kt}^{(Q_{kt})} = \dot{\underline{r}}_{2,1}^{(Q_1)} - \frac{\underline{n}_{s_1}\cdot\underline{e}_2}{\underline{g}_1\cdot\underline{e}_2} c_{kt}\,\dot{\underline{g}}_1 + c_{kt}\,\dot{\underline{n}}_{s_1} - \frac{(\underline{g}_1\cdot\underline{e}_2)(\dot{\underline{n}}_{s_1}\cdot\underline{e}_2 + \underline{n}_{s_1}\cdot\dot{\underline{e}}_2) - (\underline{n}_{s_1}\cdot\underline{e}_2)(\dot{\underline{g}}_1\cdot\underline{e}_2 + \underline{g}_1\cdot\dot{\underline{e}}_2)}{(\underline{g}_1\cdot\underline{e}_2)^2} c_{kt}\,\underline{g}_1 \qquad \ldots \quad (6.45)$$

where the dot above the vectors indicate their derivative with respect to θ_2. Since the vectors $\dot{\underline{r}}_{2,1}^{(Q_1)}$, $\dot{\underline{g}}_1$, $\dot{\underline{n}}_{s_1}$ and $\underline{g}_1$ are vectors in the $\underline{g}_1 - \underline{e}_1$ plane (refer to Section 6.2.2), the vector $\dot{\underline{r}}_{2,kt}^{(Q_{kt})}$ is in a plane parallel to the $\underline{g}_1 - \underline{e}_1$ plane. The same conclusion will be arrived at even if Eqn. (6.41) or Eqn. (6.43) is considered instead of Eqn. (6.35). Since $\underline{r}_{2,kt}^{(Q_{kt})}$ is having θ_2 as parameter, $\dot{\underline{r}}_{2,kt}^{(Q_{kt})}$ is the tangent vector at Q_{kt} to line $D_{2,kt}$.

Considering the surface $S_{1,kt}$ the position vector, $\underline{r}_{1,kt}^{(P_{kt})}$ of point P_{kt} can be obtained from Eqn. (6.10) by considering the end surface E_1. Following the reasons given in Sections 6.2.3 and 6.2.4 it can be seen that the tangent vector at P_{kt}, to line $D_{1,kt}$ is

in a plane parallel to $\underline{g}_1 - \underline{e}_1$ plane. Thus the line P_{kt} Q_{kt}, the tangent vectors at P_{kt} (to line $D_{1,kt}$) and Q_{kt} (to line $D_{2,kt}$) are all in a plane parallel to the $\underline{g}_1 - \underline{e}_1$ plane. Hence the condition for developability is satisfied. Hence P_{kt} Q_{kt} is the generatrix and lines $D_{1,kt}$ and $D_{2,kt}$ are the primary and secondary directrices for the surface $S_{1,kt}$. P_{kt} is the generic point on $D_{1,kt}$ corresponding to the point P_1 on $D_{1,1}$. Similarly Q_{kt} and Q_1 are the corresponding generic points on $D_{2,kt}$ and $D_{2,1}$ respectively.

Similarly considering Eqn. (6.37), it can be proved that the line Q_{kt} R_{kt} is the generatrix of the surface $S_{2,kt}$ corresponding to the generatrix Q_1 R_1 of the mean surface $S_{2,1}$ and that lines $D_{2,kt}$ and $D_{3,kt}$ are respectively the primary and secondary directrices for the surface $S_{2,kt}$.

It can be seen that when two equally thick surfaces are meeting along a common end surface, corresponding to a position of the generic point of the line of intersection of the mean surfaces of the two thick surfaces, the positions of the generic points on the line of intersection of other corresponding surfaces of the two thick surfaces are obtained along the line of intersection of the $\underline{g} - \underline{n}_s$ planes of the two mean surfaces.

The value of the parameter for all these points is the same.

Rest of the details is the same as for a single thick surface. For each of the thick surface, corresponding to a generatrix of the mean surface, the generatrices of other surfaces can be found and all surfaces proved to be developable.

6.5.4 Algorithm for Development of Equally Thick Surfaces in Series

A number of equally thick surfaces are in series such that

(a) any two adjacent thick surfaces meet along a common end surface as detailed in Section 6.5,

(b) the initial and final end surfaces of the thick surfaces in series is assumed to be planar and are defined and

(c) the directrices of the mean surfaces of the thick surfaces are defined.

Let there be nd number of thick surfaces in series.

Let in general

(a) TS_{kd} be a thick surface,

(b) $S_{kd,1}$ be the mean surface of the thick surface TS_{kd} ,

(c) $D_{kd,1}$ and $D_{kd+1,1}$ be respectively the primary and secondary directrices of the mean surface $S_{kd,1}$ and

(d) E_{kd} and E_{kd+1} be the end surfaces containing the directrices $D_{kd,1}$ and $D_{kd+1,1}$ respectively;

$kd = 1,2,\ldots,nd$. The thick surfaces are developed one by one as per the algorithm given below.

<u>Step 1</u> Read

(a) the thickness of the surface, h,

(b) the number of slices into which each half of a thick surface is to be divided, s_h and

(c) the number of position of the generic point to be considered along a directrix, N.

<u>Step 2</u> Calculate the distance c_{kt} of a surface $S_{kd,kt}$ from the mean surface $S_{kd,1}$; $kt = 1,2,\ldots,(2s_h+1)$.

<u>Step 3</u> Read the details of the directrix $D_{1,1}$. Also read the details of the end surface E_1.

<u>Step 4</u> Do the following steps for $kd = 1$ to nd. Set $kd = 1$.

<u>Step 5</u> If $kd > 1$, go to Step 6. If $kd = 1$, read the details of the directrix $D_{kd+1,1}$.

<u>Step 6</u> If $kd = nd$, go to Step 7. If $kd < nd$, read the details of the directrix $D_{kd+2,1}$.

<u>Step 7</u> Set kt = 1

<u>Step 8</u> Set K = 1

<u>Step 9</u> If kd > 1, go to Step 11. If kd = 1, go to Step 10.

<u>Step 10</u> (a) Fix the value of $\theta_1(K)$. From the condition for developability of the mean surface $S_{1,1}$ find the value of $\theta_2(K)$.

(b) Calculate $\underline{r}_{1,1}(K)$ and $\underline{r}_{2,1}(K)$, the position vectors of the K^{th} generic points on the directrices $D_{1,1}$ and $D_{2,1}$.

(c) Calculate $\dot{\underline{r}}_{1,1}(K)$, $\dot{\underline{r}}_{2,1}(K)$, $\ddot{\underline{r}}_{1,1}(K)$ and $\ddot{\underline{r}}_{2,1}(K)$. Also calculate the unit tangent vectors at these points.

(d) Calculate the vectors $\underline{g}_1(K)$, $\underline{n}_{s_1}(K)$ and $\underline{e}_1(K)$

(e) Calculate the vector

$$\underline{v}_1(K) = -\frac{\underline{n}_{s_1}(K) \cdot Z_{L1}}{\underline{g}_1(K) \cdot Z_{L1}} \underline{g}_1(K) + \underline{n}_{s_1}(K) \quad (6.46)$$

where Z_{L1} is the vector normal to the end surface E_1.

(f) If K > 1 , go to Step 11. If K = 1, note down the values of the vectors (refer to Section 6.3)

$$\underline{n}_{O_1} = \underline{n}_{s_1}(1)$$

$\underline{g}_{o_1} = \underline{g}_1(1)$ and

$\underline{t}_{o_1}$ = unit tangent vector to the primary directrix $D_{1,1}$.

Also calculate $\underline{e}_{o_1}$.

Step 11 (a) Calculate the arc length of the primary directrix $D_{kd,1}$ corresponding to $\theta_{kd}(K)$

$$s_{kd,1}(K) = s_{kd,1}(K-1) + \int_{\theta_{kd}(K-1)}^{\theta_{kd}(K)} [\dot{\underline{r}}_{kd,1}(K) \cdot \dot{\underline{r}}_{kd,1}(K)]^{1/2} \, d\theta_{kd} \quad .. \quad (6.47)$$

(b) Calculate the magnitude of geodesic curvature,

$$k_{g_{kd,1}}(K) = \frac{\dot{\underline{r}}_{kd,1}(K) \times \ddot{\underline{r}}_{kd,1}(K)}{[\dot{\underline{r}}_{kd,1}(K) \cdot \dot{\underline{r}}_{kd,1}(K)]^{3/2}} \cdot \underline{n}_{s_{kd}}(K) \quad ... \quad (6.48)$$

(c) Calculate the co-ordinates of the ends of the generatrix in the development

$$[x_{d_{kd,1}}(K) \, , \, y_{d_{kd,1}}(K)]$$

and

$$[x_{d_{kd+1,1}}(K), \, y_{d_{kd+1,1}}(K)]$$

Step 12 If kd = nd, go to Step 14. If kd < nd, go to Step 13.

Step 13 (a) From the condition for developability of the surface $S_{kd+1,1}$ find the value of $\theta_{kd+2}(K)$ corresponding to the value of $\theta_{kd+1}(K)$.

(b) Calculate $\underline{r}_{kd+2,1}(K)$, the position vector of the K^{th} generic point on the directrix $D_{kd+2,1}$.

(c) Calculate $\dot{\underline{r}}_{kd+2,1}(K)$, $\ddot{\underline{r}}_{kd+2,1}(K)$ and the corresponding unit tangent vector.

(d) Calculate the vectors $\underline{g}_{kd+1}(K)$, $\underline{n}_{s_{kd+1}}(K)$ and $\underline{e}_{kd+1}(K)$.

(e) Calculate the vector

$$\underline{v}_{kd+1}(K) = -\frac{\underline{n}_{s_{kd}}(K) \cdot \underline{e}_{kd+1}(K)}{\underline{g}_{kd}(K) \cdot \underline{e}_{kd+1}(K)} \underline{g}_{kd}(K) + \underline{n}_{s_{kd}}(K) \quad \ldots \quad (6.49)$$

(f) If $K > 1$, go to Step 15. If $K = 1$, note down the values of the vectors (refer to Section 6.3)

$\underline{n}^{*}_{o_1} = \underline{n}_{s_{kd+1}}(1)$,

$\underline{g}^{*}_{o_1} = \underline{g}_{kd+1}(1)$ and

$\underline{t}^{*}_{o_1}$ = unit tangent vector to the primary directrix $D_{kd+1,1}$.

Also calculate $\underline{e}^{*}_{o_1}$.

Go to Step 15.

Step 14 Calculate the vector

$$\underline{v}_{kd+1}(K) = -\frac{\underline{n}_{s_{kd}}(K).Z_{L2}}{\underline{g}_{kd}(K) \cdot Z_{L2}} \underline{g}_{kd}(K) + \underline{n}_{s_{kd}}(K) \quad \ldots \quad (6.50)$$

Where Z_{L2} is the vector normal to the end E_{nd+1}.

Step 15 If $K = N$, go to Step 16. If $K < N$, set $K = K+1$ and go to Step 9.

Step 16 Calculate the first and second derivatives of the vector $\underline{v}_{kd}(K)$, $K = 1$ to N with respect to the parameter θ_{kd}.

Step 17 The development of the surfaces S_{kt}, $kt = 2, 3, \ldots, (2s_h+1)$ of the thick surface TS_{kd} is to be carried out now. Set $K = 1$.

Step 18 Set $kt = 2$.

Step 19 Calculate the position vector of the K^{th} generic points on the directrices $D_{kd,kt}$ and $D_{kd+1,kt}$.

$$\underline{r}_{kd,kt}(K) = \underline{r}_{kd,1}(K) + c_{kt}\, \underline{v}_{kd}(K) \qquad (6.51)$$

$$\underline{r}_{kd+1,kt}(K) = \underline{r}_{kd+1,kt}(K) + c_{kt}\, \underline{v}_{kd+1}(k) \qquad (6.52)$$

Step 20 Calculate the tangent vector

$$\dot{\underline{r}}_{kd,kt}(K) = \dot{\underline{r}}_{kd,1}(K) + c_{kt}\, \dot{\underline{v}}_{kd}(K) \qquad (6.53)$$

and from this calculate the unit tangent vector. Also calculate

$$\ddot{\underline{r}}_{kd,kt}(K) = \ddot{\underline{r}}_{kd,1}(K) + c_{kt}\, \ddot{\underline{v}}_{kd}(K) \qquad (6.54)$$

Step 21 (a) Calculate the arc length

$$s_{kd,kt}(K) = s_{kd,kt}(K-1) + \int_{\theta_{kd}(K-1)}^{\theta_{kd}(K)} \dot{s}_{kd,kt}(K)\,d\theta_{kd} \quad \ldots \quad (6.55)$$

where

$$\dot{s}_{kd,kt}(K) = [\, \dot{\underline{r}}_{kd,kt} \cdot \dot{\underline{r}}_{kd,kt}]^{1/2} \quad (6.56)$$

(b) Calculate the magnitude of geodesic curvature

$$k_{g_{kd,kt}}(K) = k\underline{b}_{kd,kt}(K) \cdot \underline{n}_{s_{kd}}(K) \quad (6.57)$$

where

$$k\,\underline{b}_{kd,kt}(K) = \frac{\dot{\underline{r}}_{kd,kt}(K) \times \ddot{\underline{r}}_{kd,kt}(K)}{[\dot{s}_{kd,kt}(K)]^3} \quad (6.58)$$

Step 22 Calculate the co-ordinates of the ends of the generatrix in the development of the directrices $D_{kd,kt}$ and $D_{kd+1,kt}$

$$[x_{d_{kd,kt}}(K)\ ,\ y_{d_{kd,kt}}(K)]$$

and

$$[x_{d_{kd+1,kt}}(K)\ ,\ y_{d_{kd+1,kt}}(K)].$$

Step 23 If $K > 1$, go to Step 24. If $K = 1$, note down the value of $\underline{t}_{0_{kt}}$, the unit tangent to the directrix $D_{kd,kt}$. Also calculate the value of

$$\underline{e}_{0_{kt}} = \underline{n}_{0_1} \times \underline{t}_{0_{kt}}.$$

<u>Step 24</u> If $kt = 2s_h + 1$, go to Step 25. Otherwise set $kt = kt + 1$ and go to Step 19.

<u>Step 25</u> If $K = N$, go to Step 26. Otherwise set $K = K+1$ and go to Step 18.

<u>Step 26</u> Calculate

$$m_1 = -\frac{\underline{n}_{o_1} \cdot Z_L^*}{\underline{g}_{o_1} \cdot Z_L^*} \; \underline{g}_{o_1} \cdot \underline{t}_{o_1}$$

$$m_2 = -\frac{\underline{n}_{o_1} \cdot Z_L^*}{\underline{g}_{o_1} \cdot Z_L^*} \; \underline{g}_{o_1} \cdot \underline{e}_{o_1}$$

where

$$\begin{aligned} Z_L^* &= Z_{L1} && \text{if } kd = 1 \\ &= \underline{e}_{kd-1}(1) && \text{if } kd > 1 \end{aligned}$$

<u>Step 27</u> Set $kt = 2$.

<u>Step 28</u> Calculate

$$m_3 = \underline{t}_{o_{kt}} \cdot \underline{t}_{o_1}$$

$$m_4 = \underline{e}_{o_{kt}} \cdot \underline{t}_{o_1}$$

$$m_5 = m_1 \, c_{kt}$$

$$m_6 = \underline{t}_{o_{kt}} \cdot \underline{e}_{o_1}$$

$$m_7 = \underline{e}_{o_{kt}} \cdot \underline{e}_{o_1}$$

$$m_8 = m_2 \, c_{kt}$$

<u>Step 29</u> Set $K = 1$

Step 30 Calculate

$$x_{dm_{kd,kt}}^{(K)} = m_3\, x_{d_{kd,kt}}^{(K)} + m_4\, y_{d_{kd,kt}}^{(K)} + m_5$$

$$y_{dm_{kd,kt}}^{(K)} = m_6\, x_{d_{kd,kt}}^{(K)} + m_7\, y_{d_{kd,kt}}^{(K)} + m_8$$

$$x_{dm_{kd+1,kt}}^{(K)} = m_3\, x_{d_{kd+1,kt}}^{(K)} + m_4\, y_{d_{kd+1,kt}}^{(K)} + m_5$$

$$y_{dm_{kd+1,kt}}^{(K)} = m_6\, x_{d_{kd+1,kt}}^{(K)} + m_7\, y_{d_{kd+1,kt}}^{(K)} + m_8$$

Step 31 If K = N, go to Step 32. Otherwise set K = K+1. Go to Step 30.

Step 32 If kt = $(2s_h+1)$, go to Step 33. Otherwise set kt = kt+1 and go to Step 28.

Step 33 Calculate the following for K = 1 to N.

$$x_{dm_{kd,1}}^{(K)} = x_{d_{kd,1}}^{(K)}$$

$$y_{dm_{kd,1}}^{(K)} = y_{d_{kd,1}}^{(K)}$$

$$x_{dm_{kd+1,1}}^{(K)} = x_{d_{kd+1,1}}^{(K)}$$

$$y_{dm_{kd+1,1}}^{(K)} = y_{d_{kd+1,1}}^{(K)}$$

Step 34 If kd = nd, Stop. Otherwise set

kd = kd+1

$$\underline{n}_{o_1} = \underline{n}^{*}_{o_1}$$

$$\underline{g}_{o_1} = \underline{g}^{*}_{o_1}$$

$$\underline{t}_{o_1} = \underline{t}^{*}_{o_1}$$

$$\underline{e}_{o_1} = \underline{e}^{+x}_{o_1}$$

Go to Step 5.

6.6 Case Study

Here the development of a thick super-conical surface is presented as Example 6.1. A computer programe as per the algorithm given in Section 6.4 is written taking into account the fact that the mean surface here is a super-conical convolute. The vectors $\underline{r}_{i,1}$, $\dot{\underline{r}}_{i,1}$ and $\ddot{\underline{r}}_{i,1}$ are given by Eqns. (3.7), (3.8) and (3.12) respectively (refer to Section 3.4.1). Subsituting for $\dot{\underline{r}}_{i,1}$ in Eqn. (6.24) and rearranging the terms, the following expression is obtained for $\dot{s}_{i,kt}$.

$$\dot{s}_{i,kt} = \left(\frac{2}{n_i}\right) \cos^{(2/n_i-1)}\theta_i \ \sin^{(2/n_i-1)}\theta_i \ (\text{Factors})^{1/2} \qquad \ldots \qquad (6.59)$$

where

$$(\text{Factors}) = [(\text{Factor } 1)_i^2 + 2c_{kt}\left(\frac{n_i}{2}\right) \cos^{(1-2/n_i)}\theta_i \ \sin^{(1-2/n_i)}\theta_i (\text{Factor2})_i^2 + c_{kt}^2\left(\frac{n_i}{2}\right)^2 \cos^{(2-4/n_i)}\theta_i \ \sin^{(2-4/n_i)}\theta_i (\text{Factor3})_i^2]. \qquad \ldots \qquad (6.60)$$

The term $(\text{Factor } 1)_i^2$ is given by Eqn. (3.11) (refer to Section 3.4.1) and

$$(\text{Factor } 2)_i^2 = \sum_{k=1}^{3} \{ - t_{i_{k,1}} \, a_i \sin^{(2-2/n_i)} \theta_i$$

$$+ t_{i_{k,2}} \, b_i \cos^{(2-2/n_i)} \theta_i \} \, \dot{v}_{i_k} \, .$$

$$\ldots \qquad (6.61)$$

$$(\text{Factor } 3)_i^2 = \dot{\underline{v}}_i \cdot \dot{\underline{v}}_i$$

where $\quad \dot{\underline{v}}_i = (\dot{v}_{i_1} \;\; \dot{v}_{i_2} \;\; \dot{v}_{i_3})$.

For $n > 2$, when $\cos \theta_i$ or $\sin \theta_i$ equals zero, the vector $\underline{f}_{i,kt} \rightarrow \underline{f}_{i,1}$ since $\underline{f}_{i,1} \rightarrow \infty$.

In a similar manner the expression for the $k\underline{b}$ vector is obtained as

$$k\underline{b} = \frac{\sin^{(1-4/n_i)} \theta_i \cos^{(1-4/n_i)} \theta_i}{(\text{Factors})^{3/2}} \, (\underline{\varepsilon}) \qquad (6.62)$$

where

$$\underline{\varepsilon} = (n_i - 1) \, a_i \, b_i \sin \theta_i \cos \theta_i \; \underline{\varepsilon}_1 \times \underline{\varepsilon}_2$$

$$+ c_{kt} \left(\frac{n_i}{2}\right)^2 \sin \theta_i \cos \theta_i \, [a_i \sin^{(2-2/n_i)} \theta_i \; \dot{\underline{v}}_i \times \underline{\varepsilon}_1$$

$$+ b_i \cos^{(2-2/n_i)} \theta_i \; \underline{\varepsilon}_2 \times \ddot{\underline{v}}_i]$$

$$+ c_{kt} \left(\frac{n_i}{2}\right)^2 [(2/n_i \sin^2 \theta_i - 1) \, a_i \sin^{(2-2/n_i)} \theta_i \; \dot{\underline{v}}_i \times \underline{\varepsilon}_1$$

$$+ \left(\frac{2}{n_i} \cos^2 \theta_i - 1\right) b_i \cos^{(2-2/n_i)} \theta_i \; \dot{\underline{v}}_i \times \underline{\varepsilon}_2]$$

$$+ c_{kt}^2 \left(\frac{n_i}{2}\right)^3 \sin^{(2-2/n_i)} \theta_i \cos^{(2-2/n_i)} \theta_i \; \dot{\underline{v}}_i \times \ddot{\underline{v}}_i \, .$$

$$(6.63)$$

Here

$$\underset{\sim}{\varepsilon}_1 = (t_{i_{1,1}}\ t_{i_{2,1}}\ t_{i_{3,1}})$$

$$\underset{\sim}{\varepsilon}_2 = (t_{i_{1,2}}\ t_{i_{2,2}}\ t_{i_{3,2}}) .$$

When $\cos\theta_i$ or $\sin\theta_i$ equals zero, for $n_i > 2$, the curvature vector is obtained from Eqn. (3.18), since $\dot{r}_{i,1} \rightarrow \infty$.

6.6.1 <u>Example 6.1</u>

The input data about the geometry of the mean surface is given below

Primary directrix:

$a_i = 0.9$

$b_i = 0.58$

$n_i = 3.1$

$\gamma_i = 0.0$

$$[MTH_i] = \begin{bmatrix} -0.463 & -0.835 & 0.293 & 4.72 \\ 0.605 & -0.052 & 0.794 & 8.52 \\ -0.648 & 0.548 & 0.530 & 4.48 \\ 0.0 & 0.0 & 0.0 & 1 \end{bmatrix}$$

Secondary directrix:

$a_j = 1.0$

$b_j = 0.6$

$n_j = 3.2$

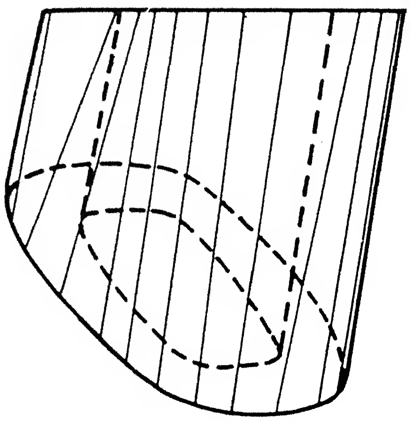
$\gamma_j = 0.0$

$$[MTH_j] = \begin{bmatrix} -0.447 & -0.894 & 0.0 & 5.0 \\ 0.0 & 0.0 & 1.0 & 5.0 \\ -0.894 & 0.447 & 0.0 & 5.0 \\ 0.0 & 0.0 & 0.0 & 1 \end{bmatrix}$$

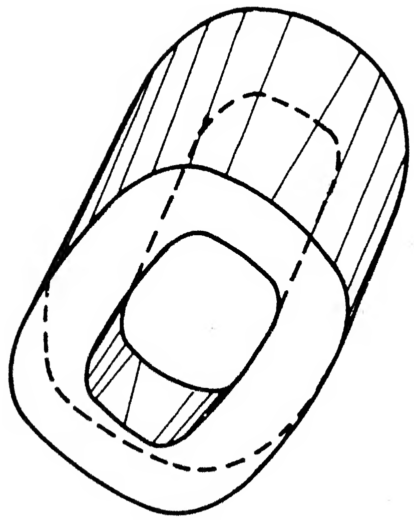
Thickness of surface, h = 0.4

Number of slices into which each half of the thick surface is to be divided, s_h = 2

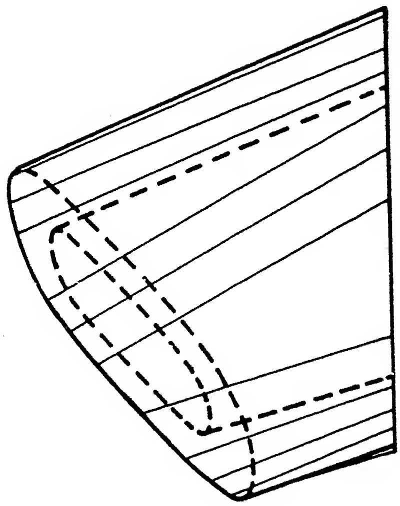
The orthographic views of the thick surface is shown in Fig. 6.8a. The development of the thick surface is given in Fig. 6.8b.



TOP VIEW



FRONT VIEW



SIDE VIEW

Fig 6.8a Orthographic views of thick surface. Example 6.1.

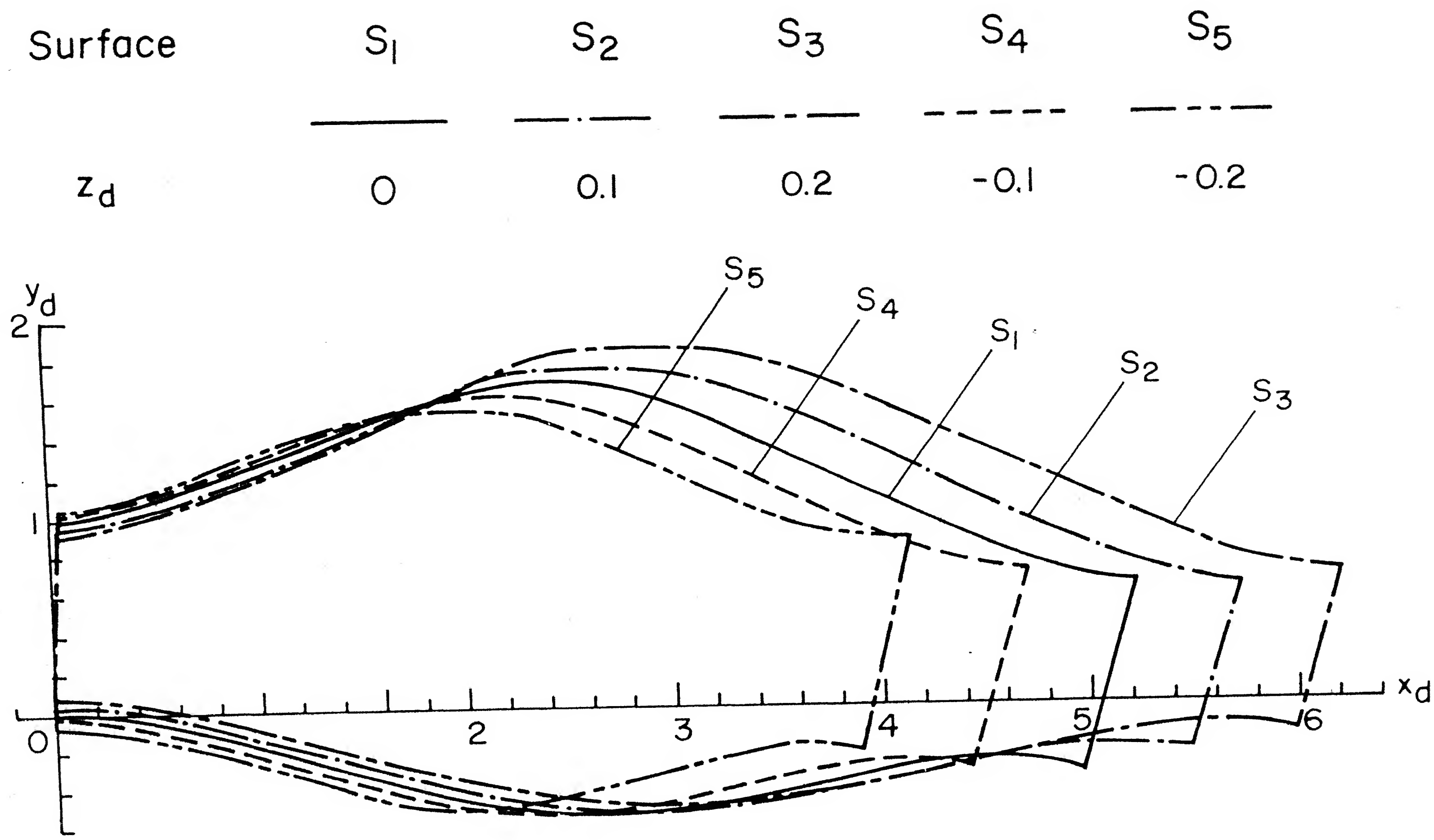


Fig. 6.8b Development of thick surface. Example 6.1.

Chapter 7

CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

The mathematical aspects about development of single curved ruled surfaces either with a single directrix or with two directrices are studied in Chapter 2 and suitable algorithms for the development of these types of surfaces are given in a general manner [32]. In subsequent chapters the development of super conical convolutes, helical convolutes, thin ducts and thick surfaces are discussed in detail. Suitable algorithms and few case studies have been given in each chapter.

7.1 Conceptual Aspects of the Present Work

In Chapter 3 the development of super conical convolutes is given. The method can be used for a ruled surface of general form also. If the edges E_1 and E_2 of the surface to be developed are not rulings of the surface (refer to Figure 7.1) then the portion of the surface between these edges and the nearest ruling (shown shaded in the figure) can be developed by triangulation method. Or the edge and the portion of the directrix forming the two sides of the portion of the surface can be treated

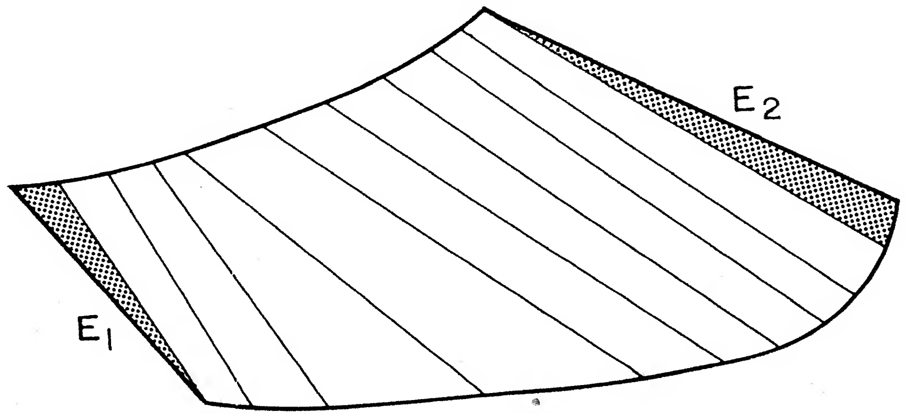


Fig. 7.1 Edges E_1 and E_2 are not rullings of the surface.

as the directrices and the rulings of the portion of the surface obtained. Then the development can be carried out as usual.

The development of helical convolutes with cylindrical or conical helix of either a circular or an elliptical base is given in Chapter 4. Super-ellipses with $n \gtrless 2$ (refer to Figure 3.2) can not be used as base for the helix of the convolute. If $n < 2$, then corresponding to $\theta = 0, \pi/2, \pi$ and $3\pi/2$ there is discontinuity in the tangent to the super-ellipse and hence in the tangent to the helix using the super-ellipse as the base. So there will be discontinuity in the surface of the helical convolute. If $n > 2$, then around $\theta = 0, \pi/2, \pi$ and $3\pi/2$ there are straight line portions in the super-ellipse. Hence the tangent to the helix will coincide at these portions. This will result in the surface degenerating to a line at these portions.

In Chapter 5, it is shown that a thin duct can be approximated to a set of super-conical convolutes in series. It is shown that the spatial configuration and shape and size of these super conical convolutes can be expressed as functions of the parameter of the centre-line of the duct. Development of the duct is achieved by developing these super-conical convolutes individually.

This method can be used for developing approximately double curved surfaces. Duct is nothing but a double curved surface of general form. The constant or variable curve and the general curved path along which it moves to generate the double curved surface correspond respectively to the curve of cross section and the centre line of the duct. In the case of double curved surfaces generated by a curve revolving about an axis, the axis of revolution correspond to the centre line. Here the surface is approximated to a number of frusta of cone in series. As stated in Chapter 5, depending upon the accuracy desired and the complexity of the geometry of the surface (which in turn depends upon the nature of the centre line and the variation in shape and size of the curved generatrix), the number and locations of the cross section of the surface can be chosen.

Uniformly thick surfaces are considered for development in Chapter 6. A thick surface is considered to be a set of thin surfaces lying one over the other. A mean surface is defined. It should be developable. This is the main criteria for the development of the thick surface. Once the mean surface is developable, other surfaces in the set are shown to be developable. They are defined in terms of the mean surface and the distance

between them and the mean surface. These surfaces are developed individually and then a co-ordinate transformation is applied to these developments to take into account the fact that the origin and axes of the development are different for different surfaces. This completes the development of thick surface.

The case of two equally thick surfaces in series, i.e. meeting along a common end surface, is also discussed in Chapter 6. The mean surfaces of these two thick surfaces are to be developable. Mathematical expressions for defining the points on the common end surface are given. Finally suitable algorithm to develope multiple thick surfaces in series is given. This method can be used to develope a thick surface whose mean surface is as a whole not developable, but piecewise developable. Then the mean surface can be divided into a number of surface patches, each one of which is individually developable. The corresponding portions of the thick surface constitute a set of thick surfaces in series and each one of them can be developed individually [33].

7.2 Numerical Aspects of the Present Work

Based on the algorithm given in each chapter, computer programmes have been developed and used. They are as follows:

(i) a programme for the development of super-conical convolute when the space configuration of the convolute is given as per definition (A) (refer to Sec. 3.3.1),

(ii) a programme for the development of super-conical convolute when the space configuration of the convolute is given as per definition (B) (refer to Sec. 3.3.2),

(iii) a programme for the development of cylindrical helical convolute with circular base,

(iv) a programme for the development of conical helical convolute with circular base,

(v) a programme for the development of conical helical convolute with elliptical base,

(vi) a programme for the development of thin ducts,

(vii) a programme for the development of a thick surface,

(viii) a programme for the development of multiple thick surfaces in series and

(ix) a programme for finding the angular parameter β of the direction vector of the generatrix of a single curved ruled surface with one directrix (refer to Sec. 2.3.2).

The programme used for the development of conical helical convolutes with elliptical base is the most general of the three programmes developed for the development of helical convolutes. If $a_{i0} = b_{i0}$, the base of the helix is a circle and if $e_i = 0$, the helix is a cylindrical one.

All these programmes are in FORTRAN. Suitable modules have been developed for performing certain computations like finding the root of the algebraic equation that, in turn, is the condition for developability, the calculation of arc length, the calculation of arc-tangent angle etc. and these are used in various programmes whenever such computations are required. Suitable graphics module using GPGS are used for displaying (a) the orthographic views of the surface in third angle projection (b) the development of the surface and (c) the graph of the magnitude of geodesic curvature versus arc length [34]. These programmes were run on DEC-1090 system and the results obtained have been discussed in Chapters 2 to 6.

Wherever necessary, suitable numerical techniques are used in these programmes. Bisection method is used for finding the root of the algebraic equation of condition for developability (in the case of ruled surfaces,

class II). Fourth-order Runge-Kutta scheme [35] is used for integration of Serret-Frenet equations and also for the calculation of the arc-tangent angle. This scheme is also used for the calculation of the arc length in the case of helical convolutes. In the case of super-conical convolutes the arc length is calculated using a NAG routine available in the system [36]. This is to cope up with the integrand blowing up at certain values of the parameter of the arc (the primary directrix).

In the case of helical convolutes the magnitude of geodesic curvature, the arc length and the arc-tangent angle are all expressed as functions of the parameter of the helix. Also the Serret-Frenet equations are expressed in terms of this parameter. Because of this fact there is no loss of accuracy during interpolation used in the fourth order Runge-Kutta scheme for the integration of Serret-Frenet equations etc.

The time taken by the individual programmes depends to a larger extent on the time taken by the bisection method. This, in turn, depends upon (a) the step length used to trap the interval in which the root lies and (b) the accuracy with which the root is found out. The CPU time taken by the various programmes for the case studies presented in the Chapters 3 to 6 are given in Table 7.1. These correspond to a step length = 0.001,

and to an accuracy = 0.002 used to find the root of the condition for developability.

7.3 Suggestions for Further Work

In Chapter 6 the development of a uniformly thick surface is carried out by treating it to be a set of thin surfaces lying one over the other. These thin surfaces are developed individually and the developments are stacked together to get the development of the thick surface. Since the surfaces are thin and are developed individually treating them to be independent of one another, during this process, no plastic deformation is considered to occur during the manufacturing processes of folding and bending to form the surface out of a plane sheet. Thus the development of the thick surface obtained here does not take into account the plastic deformations.

Suitable modification to take into account the plastic deformations is to be applied to the development obtained here so that the final development of the curved surface is accurate. A thick plane sheet (of the specified material) cut to the shape and size of the final development should give the required curved thick surface when subjected to the specified manufacturing processes. This is the final aim of the development of thick surfaces. Suitable mathematical model and algorithms need

to be developed for the modification of the development obtained here to get the final development.

In case only one piece of the surface is to be fabricated, then an optimum rectangle enclosing the development of the surface is to be found out such that the wastage of material is minimum. Then the co-ordinates of the points on the development are to be given with respect to the two adjacent edges of the rectangle. An algorithm for this purpose is given by Dhande and Ramulu [27]. A programme module developed as per this algorithm can be called by the main programme once the development is achieved.

In case more than one piece of the surface is to be fabricated, then depending upon the size of the sheet metal available, the developments are to be nested so that wastage of material is minimum. Suitable methods to solve this problem and an algorithm need to be worked out. A programme module written as per this algorithm can then be called for by the main programme to express the coordinates of points on the development with respect to the two adjacent edges of the rectangular sheet metal used.

Once these problems are solved and the final data of the development is obtained, this can be transferred

to a numerical controlled machine for automated production. Interactive design packages to take care of all these aspects need to be developed as a part of CAD software system.

Development process finds application in the fabrication of dress. Mathematical models are to be developed for the fabrication of different types of dress like a shirt, a trouser, a coat etc. These mathematical models should take into account the shape of the human body, the relative positions of the parts of the body at different postures, the amount of comfort required at various postures, the aesthetic view point, the flexibility of the dress material and the requirements of the seams connecting together the various parts of the dress. Once the surface of the dress is finalized, it can be split into a set of developable surfaces. If this is not possible, then the surface of the dress can be approximated to a set of developable surfaces, and the inaccuracy involved in the approximation can to some extend be offset by the flexibility of the dress material. Finally when mass production is involved, the nesting problem needs to be solved taking into account the direction-orientation property of the strength of the dress material.

Table 7.1 The CPU Time of Examples Given

Example No.	CPU time, Min : Sec
3.1	0 : 1.20
3.2	0 : 2.75
3.3	0 : 7.36
3.4	0 :21.47
4.1	0 : 0.33
4.2	0 : 0.28
4.3	0 : 0.44
5.1	0 : 7.53
5.2	0 :50.61
6.1	0 :15.43

REFERENCES

1. Barnhill, Robert E. and Riesenfeld, Richard F. (Ed), Computer Aided Geometric Design, Academic Press, New York, 1974.

2. French, Thomas E., Vierck, Charles J. and Foster, Robert J., Graphic Science and Design, Fourth Ed., McGraw-Hill, New York, 1984.

3. Hood, George J., Palmerlee, Albert S. and Baer, Charles J., Geometry of Engineering Drawing, Fifth Ed., McGraw-Hill, New York, 1969.

4. Rule, J.T. and Coons, S.A., Graphics, McGraw-Hill, New York, 1961.

5. Street, W.E., Technical Descriptive Geometry, Sixth Ed., D. Van Nostrand, New Jersey, London, 1959.

6. Luzadder, W.J., Graphics for Engineers, Prentice-Hall, Englewood Cliffs, New Jersey, 1958.

7. Hawk, M.C., Theory and Problems of Descriptive Geometry, Schaum Series, McGraw-Hill, New York, 1962.

8. Rowe, C.E. and McFarland, J.D., Engineering Descriptive Geometry, D. Van Nostrand, New Jersey, London, 1961.

9. Newmann, William M. and Sproull, Robert F., Principles of Interactive Computer Graphics, Second Ed., McGraw-Hill, Tokyo, 1981.

10. Harrington, Stevan, Computer Graphics - A Programming Approach, McGraw-Hill, Tokyo, 1983.

11. Foley, James D. and Van Dam, Andries, Fundamentals of Interactive Computer Graphics, Addison-Wesley, Reading, Massachusetts, 1984.

12. Faux, I.D. and Pratt, M.J., Computational Geometry for Design and Manufacture, Ellis Horwood, Chichester, 1983.

13. Rogers, D.F. and Adams, J.A., Mathematical Elements for Computer Graphics, McGraw-Hill, New York, 1976.

14. Encarnacao, J . and Schlechtendahl, E.G., Computer Aided Design, Springer-Verlag, Berlin, 1983.

15. Lord, E.A. and Wilson, C.B., The Mathematical Description of Shape and Form, Ellis Horwood, Chichester, 1984.

16. Baer, A., Eastman, C. and Henrion, M., " Geometric Modelling : a Survey", Computer-aided Design, Vol. 11, No. 5, Sep 1979, pp. 253-272.

17. Goetz, Abraham,, Introduction to Differential Geometry Addison-Wesley, Reading, Massachusetts, 1970.

18. Struik, Dirk J., Lectures on Classical Differential Geometry, Second Ed., Addison-Wesley, Reading, Massachusetts, 1961.

19. Willmore, T. J., An Introduction to Differential Geometry, Oxford University, 1966.

20. Kreyszig, Erwin,, Differential Geometry, University of Toronto, Toronto, 1959.

21. Pogorelov, A.V., Differential Geometry, translated by Boron, Leo F., P.Noordhooff N.V., Groningen, the Netherlands.

22. Lipschutz, Martin, M., Theory and Problems of Differential Geometry, Schaum Series, McGraw-Hill, New York, 1969.

23. Hilbert D. and Cohn-Vossen, S., Geometry and the Imagination, translated by Nemenyi, P., Chelsea, New York, 1952.

24. Prakash, Nirmala., Differential Geometry-An Integrated Approach, Tata McGraw-Hill, New Delhi 1981.

25. Betterley, Melvin L., Sheet Metal Drafting, McGraw-Hill, New York, 1961.

26. Manning, J.R., " Computerized Pattern Cutting - Methods based on an Isometric Tree", Computer Aided Design, Vol. 12, No. 1, Jan 1980, PP.43-47.

27. Dhande, S.G. and Ramulu, P.B., " Computer Aided Methods for Development of Transition Sections", Journal of Mechanisms, Transmission and Automation in Design, Transactions of the ASME, Vol. 106, Sep 1984, PP. 401-408.

28. Fournier, A. and Wesley, M.A., " Bending Polyhedral Objects", Computer Aided Design, Vol. 15, No. 2, Mar 1983, PP. 79-87.

29. Ralston, Anthony, A First Course in Numerical Analysis, McGraw-Hill, New York, 1965.

30. Pennington, Ralph H., Introductory Computer Methods and Numerical Analysis, Collier Mac Millan, London, 1971.

31. Flanagen, D.L. and Hefner, O.V., " Surface Moulding-New Tool for the Engineer", Aeronautics and Astronautics, Apr 1967, pp. 58-62.

32. Gurunathan, B. and Dhande, S.G., " Algorithms for Development of Certain Classes of Ruled Surfaces", submitted for publication in the journal Computers and Graphics.

33. Gurunathan, B. and Dhande, S.G., " A Computer Aided Geometric Method for Development of Thick-Surface Ducts", in Preparation for submission to the Design Automation Committee of ASME, USA for presentation at the Design Engineering Technical Conference, 1986.

34. Mudur, S.P., General Purpose Graphics Systems (GPGS) Manual, Second Ed., NCSDCT, TIFR, Bombay, 1977.

35. Krishnamurthy, E.V. and Sen, S.K., Computer Based Numerical Algorithms, Affiliated East West, New Delhi, 1976.

36. NAG Fortran Library Manual, Mark 8, Vol. 1, 1981.

Appendix I

SIGN/VALUE OF FRACTIONAL POWERS

In the calculation of various quantities like the position vector of a generic point of the directrix, the tangent vector, the arc length, the curvature etc. and also in the condition for developability of super-conical convolutes, calculation of fractional powers of $\cos\theta_i$, $\sin\theta_i$, $\cos\theta_j$ and $\sin\theta_j$ are involved. Depending upon the value of θ_i and θ_j these terms $\cos\theta_i$, $\sin\theta_i$, $\cos\theta_j$ and $\sin\theta_j$ take either positive or negative values. So calculation of fractional powers of quantities which are either positive or negative is involved. This is done by taking into account the geometrical requirements as detailed below.

I.1 Calculation of Position Vector

The position vector of a generic point of a super-ellipse is given by

$$\underline{r} = \begin{bmatrix} a\cos^{2/n}\theta \\ b\sin^{2/n}\theta \\ 0 \\ 1 \end{bmatrix} \qquad (I.1)$$

From the gemetry of the super-ellipse the sign of $\cos^{2/n}\theta$ is fixed to be positive when $\cos\theta$ is positive and negative when $\cos\theta$ is negative. The value of $\cos^{2/n}\theta$ is fixed to be 1 when the value of $\cos\theta$ is 1 and the value is fixed to be -1 when $\cos\theta$ is -1. Similarly the sign/value of $\sin^{2/n}\theta$ is fixed according to the sign/value of $\sin\theta$ (refer to Figure I.1 and Table I.1).

I.2 Calculation of Tangent and Unit Tangent Vectors

The tangent vector and the unit tangent vectors to the super-ellipse are given by

$$\dot{\underline{r}} = \frac{2}{n}\cos^{(2/n-1)}\theta\,\sin^{(2/n-1)}\theta \begin{bmatrix} -a\sin^{(2-2/n)}\theta \\ b\cos^{(2-2/n)}\theta \\ 0 \end{bmatrix} \quad \ldots \quad (I.2)$$

and

$$\underline{t} = \frac{1}{(\text{Factor } 1)} \begin{bmatrix} -a\sin^{(2-2/n)}\theta \\ b\cos^{(2-2/n)}\theta \\ 0 \end{bmatrix} \quad (I.3)$$

where

$$(\text{Factor } 1) = [\{-a\sin^{(2-2/n)}\theta\}^2 + \{b\cos^{(2-2/n)}\theta\}^2]^{1/2} \quad (I.4)$$

(Factor 1) is always positive and considering the direction of the vector $\underline{t}$, the sign/value of the terms $\cos^{(2-2/n)}\theta$ and $\sin^{(2-2/n)}\theta$ are fixed. Also the sign/value of the terms $\cos^{(2/n-1)}\theta$ and $\sin^{(2/n-1)}\theta$ are fixed accordingly (refer to Figure I.1 and Table I.1).

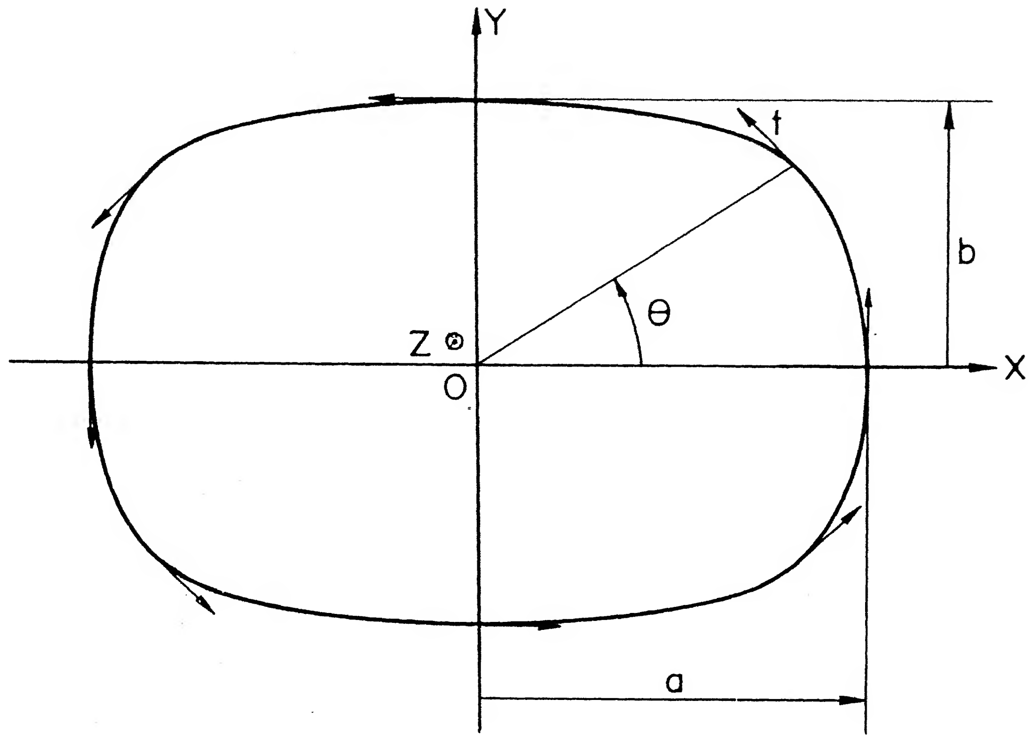


Fig.I.1. Tangent to super-ellipse.

I.3 The Second Derivative of Position Vector

The second derivative of position vector is given by

$$\ddot{\underline{r}} = \frac{2}{n} \cos^{(2/n-2)}\theta \sin^{(2/n-2)}\theta \begin{bmatrix} (2/n \sin^2\theta - 1)a \sin^{(2-2/n)}\theta \\ (2/n \cos^2\theta - 1)b \cos^{(2-2/n)}\theta \\ 0 \end{bmatrix} \quad \ldots (I.5)$$

The sign/value fixed for the terms $\sin^{(2-2/n)}\theta$ and $\cos^{(2-2/n)}\theta$ satisfy the geometrical requirement.

I.4 Other Quantities

Other quantities like curvature, arc length etc. and also the condition for developability are functions of $\underline{r}$, $\dot{\underline{r}}$ and $\ddot{\underline{r}}$. Hence the above mentioned sign/values fixed for the various terms like $\cos^{2/n}\theta$, $\cos^{(2-2/n)}\theta$, $\cos^{(2/n-1)}\theta$ etc. satisfy the requirements.

Table I.1 Sign/Value of Fractional Powers

	Quadrant 1	2	3	4	θ, deg. 0	90	180	270
$\cos\theta$	+ ve	− ve	− ve	+ ve	1	0	−1	0
$\cos^{2/n}\theta$	+ ve	− ve	− ve	+ ve	1	0	−1	0
$\cos^{(2-2/n)}\theta$	+ ve	− ve	− ve	+ ve	1	0	−1	0
$\cos^{(2/n-1)}\theta$	+ ve	+ ve	+ ve	+ ve	1	$\infty(n>2)$ $1(n=2)$	1	$\infty(n>2)$ $1(n=2)$
$\cos^{(2-4/n)}\theta$	+ ve	+ ve	+ ve	+ ve	1	$0(n>2)$ $1(n=2)$	1	$0(n>2)$ $1(n=2)$
$\sin\theta$	+ ve	+ ve	− ve	− ve	0	1	0	−1
$\sin^{2/n}\theta$	+ ve	+ ve	− ve	− ve	0	1	0	−1
$\sin^{(2-2/n)}\theta$	+ ve	+ ve	− ve	− ve	0	1	0	−1
$\sin^{(2/n-1)}\theta$	+ ve	+ ve	+ ve	+ ve	$\infty(n>2)$ $1(n=2)$	1	$\infty(n>2)$ $1(n=2)$	1
$\sin^{(2-4/n)}\theta$	+ ve	+ ve	+ ve	+ ve	$0(n>2)$ $1(n=2)$	1	$0(n>2)$ $1(n=2)$	1

Appendix II

INTEGRATION OF SERRET-FRENET EQUATIONS

The Serret-Frenet equations are given in Chapter 2 as Eqns. (2.21) and are reproduced below.

$$\frac{d^2x}{ds^2} + k_g(s)\,\frac{dy}{ds} = 0$$

$$\frac{d^2y}{ds^2} - k_g(s)\,\frac{dx}{ds} = 0$$

If the geodesic curvature $k_g(s)$ and the arc length s are continuous functions of a parameter θ then the Serret-Frenet equations can be rewritten in terms of the parameter θ.

Let

$$\frac{ds}{d\theta} = f_1(\theta)$$

and

$$k_g(s) = f_2(\theta) \qquad \text{(II.1)}$$

then

$$\frac{dx}{ds} = \frac{1}{f_1(\theta)}\,\frac{dx}{d\theta}$$

$$\frac{dy}{ds} = \frac{1}{f_1(\theta)}\,\frac{dy}{d\theta}$$

$$\frac{d^2x}{ds^2} = \frac{1}{\{f_1(\theta)\}^2}\left[\frac{d^2x}{d\theta^2} - \frac{f_3(\theta)}{f_1(\theta)}\,\frac{dx}{d\theta}\right] \qquad \text{(II.2)}$$

and

$$\frac{d^2y}{ds^2} = \frac{1}{\{f_1(\theta)\}^2}\left[\frac{d^2y}{d\theta^2} - \frac{f_3(\theta)}{f_1(\theta)}\,\frac{dy}{d\theta}\right]$$

where

$$f_3(\theta) = \frac{d}{d\theta} f_1(\theta) \quad . \tag{II.3}$$

Substituting Eqns. (II.1) and (II.2) into Serret-Frenet equations and simplifying, the Serret-Frenet equations reduce to

$$\begin{aligned} \frac{d^2x}{d\theta^2} - \frac{f_3(\theta)}{f_1(\theta)} \frac{dx}{d\theta} + f_1(\theta)\, f_2(\theta) \frac{dy}{d\theta} &= 0 \\ \frac{d^2y}{d\theta^2} - \frac{f_3(\theta)}{f_1(\theta)} \frac{dy}{d\theta} - f_1(\theta)\, f_2(\theta) \frac{dx}{d\theta} &= 0 \quad . \end{aligned} \tag{II.4}$$

Let $y_1 = x$,

$y_2 = dx/d\theta$,

$y_3 = y$,

and $y_4 = dy/d\theta$.

Then from Eqns. (II.4) and (II.5)

$$\begin{aligned} \frac{dy_1}{d\theta} &= y_2 \\ \frac{dy_2}{d\theta} &= \frac{f_3(\theta)}{f_1(\theta)} y_2 - f_1(\theta)\, f_2(\theta)\, y_4 \\ \frac{dy_3}{d\theta} &= y_4 \\ \frac{dy_4}{d\theta} &= \frac{f_3(\theta)}{f_1(\theta)} y_4 + f_1(\theta)\, f_2(\theta)\, y_2 \quad . \end{aligned} \tag{II.6}$$

Integrating Eqns. (II.6) values of x and y can be found for various values of the parameter θ.

The initial conditions are

$$x = 0$$

$$\frac{dx}{ds} = 1$$

$$y = 0 \tag{II.7}$$

and $$\frac{dy}{ds} = 0 \quad .$$

In terms of the parameter θ, the initial conditions are

$$x = 0$$

$$\frac{dx}{d\theta} = f_1(\theta_0) \tag{II.8}$$

$$y = 0$$

and $$\frac{dy}{d\theta} = 0$$

where θ_0 is the initial value of θ.

Suitable numerical method can be used for integration of Eqns. (II.6). In this work Fourth order Runge-Kutta method is used.